

# Fully Dynamic Maximal Independent Set with Sublinear in $n$ Update Time

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## Abstract

The first fully dynamic algorithm for maintaining a maximal independent set (MIS) with update time that is *sublinear in the number of edges* was presented recently by the authors of this paper [Assadi *et al.*, STOC'18]. The algorithm is deterministic and its update time is  $O(m^{3/4})$ , where  $m$  is the (dynamically changing) number of edges. Subsequently, Gupta and Khan and independently Du and Zhang [arXiv, April 2018] presented deterministic algorithms for dynamic MIS with update times of  $O(m^{2/3})$  and  $O(m^{2/3}\sqrt{\log m})$ , respectively. Du and Zhang also gave a randomized algorithm with update time  $O(\sqrt{m})$ . Moreover, they provided some partial (conditional) hardness results hinting that the update time of  $m^{1/2-\varepsilon}$ , and in particular  $n^{1-\varepsilon}$  for  $n$ -vertex dense graphs, is a natural barrier for this problem for any constant  $\varepsilon > 0$ , for deterministic and randomized algorithms that satisfy a certain natural property.

In this paper, we break this natural barrier and present the first fully dynamic (randomized) algorithm for maintaining an MIS with update time that is always *sublinear in the number of vertices*, namely, an  $\tilde{O}(\sqrt{n})$  expected amortized update. We also show that a simpler variant of our algorithm can already achieve an  $\tilde{O}(m^{1/3})$  expected amortized update time, which results in an improved performance over our  $\tilde{O}(\sqrt{n})$  update time algorithm for sufficiently sparse graphs, and breaks the  $m^{1/2}$  barrier of Du and Zhang for all values of  $m$ .

## 1 Introduction

The maximal independent set (MIS) problem is a fundamental problem in graph theory with a wide range of applications. MIS is often used in the context of graph coloring, as all vertices in an independent set

can be assigned the same color. As another example, Hopcroft and Karp [10] gave an algorithm to compute a large bipartite matching (approximating the maximum matching to within a factor arbitrarily close to 1) by finding maximal independent sets of longer and longer augmenting paths. In general, the MIS problem has natural connections to various important combinatorial optimization problems; see the celebrated papers of Luby [18] and Linial [17] for some basic applications. Additional applications of MIS include leader election [6], resource allocation [24], network backbone constructions [11, 14], and sublinear-time approximation algorithms [21].

The MIS problem has been extensively studied in parallel and distributed settings, following the seminal works of [2, 17, 18]. Surprisingly however, the fundamental problem of *maintaining* an MIS in dynamic graphs received no attention in the literature until the pioneering PODC'16 paper of Censor-Hillel, Haramaty, and Karnin [5], who developed a randomized algorithm for this problem under the oblivious adversarial model<sup>1</sup> in *distributed* dynamic networks. Implementing the distributed algorithm of [5] in the sequential setting requires  $\Omega(\Delta)$  update time in *expectation*, where  $\Delta$  is a fixed upper bound on the maximum degree in the graph, which may be  $\Theta(m)$  even in sparse graphs. Furthermore, it is unclear whether  $O(\Delta)$  time is also sufficient for this algorithm, and a naive implementation may incur an update time of  $\Theta(m)$ , even in expectation, where  $m$  is the (dynamically changing) number of edges; see Section 6 of [5] for further details.

We study the MIS problem in a (sequential) *dynamic setting*, where the underlying graph evolves over time via edge updates. A *dynamic graph* is a graph sequence  $\mathcal{G} = (G_0, G_1, \dots, G_M)$  on  $n$  fixed vertices, where the initial graph is  $G_0 = (V, \emptyset)$  and each graph  $G_i = (V, E_i)$  is obtained from the previous graph  $G_{i-1}$  in the sequence by either adding or deleting a single edge. The work of Censor-Hillel

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<sup>1</sup>In the standard *oblivious adversarial model* (cf. [4, 12]), the adversary knows all the edges in the graph and their arrival order, as well as the algorithm to be used, but is unaware of the random bits used by the algorithm, and so cannot choose updates adaptively in response to the randomly guided choices of the algorithm.

*et al.* [5] left the following question open: Can one dynamically maintain an MIS in time significantly lower than it takes to recompute it from scratch following every edge update?

The authors of this paper answered this question in the affirmative in [3] by presenting the first fully dynamic algorithm for maintaining an MIS with (amortized) update time that is sublinear in the number of edges, namely,  $O(\min\{m^{3/4}, \Delta\})$ . Achieving an update time of  $O(\Delta)$  is simple, and the main contribution of [3] is in further reducing the update time to  $O(m^{3/4})$ . Note that  $O(m^{3/4})$  improves over the simple  $O(\Delta) = O(n)$  bound only for sufficiently sparse graphs.

Onak *et al.* [22] studied “uniformly sparse” graphs, as opposed to the work by Assadi *et al.* [3], which focused on unrestricted sparse graphs. The “uniform sparsity” of the graph is often measured by its *arboricity* [19, 20, 23]. A dynamic graph of *arboricity*  $\alpha$  is a dynamic graph such that all graphs  $G_i$  have arboricity bounded by  $\alpha$ . Onak *et al.* [22] showed that for any dynamic  $n$ -vertex graph of arboricity  $\alpha$ , an MIS can be maintained with amortized update time  $O(\alpha^2 \log^2 n)$ , which reduces to  $O(\log^2 n)$  in constant arboricity graphs, such as planar graphs. The result of [22] improves that of [3] for all graphs with arboricity bounded by  $m^{3/8-\epsilon}$ , for any constant  $\epsilon > 0$ . Nonetheless, for general graphs, this update time of  $O(\alpha^2 \log^2 n)$  is in fact higher than the naive  $O(m)$  time needed to compute an MIS from scratch (arboricity can be as high as  $O(\sqrt{m})$  in arbitrary graphs).

Recently, the  $O(m^{3/4})$  bound of Assadi *et al.* [3] for general graphs was improved to  $O(m^{2/3})$  by Gupta and Khan [8] and independently to  $O(m^{2/3} \sqrt{\log m})$  by Du and Zhang [7]. All the aforementioned algorithms (besides the distributed algorithm of [5]) are deterministic. Du and Zhang also presented a randomized algorithm under the oblivious adversarial model with an *expected* update time of  $O(\sqrt{m} \log^{1.5} m)$ ; for dense graphs, this update time reduces to  $O(n \log^{1.5} n)$ , which is worse than the simple  $O(\Delta) = O(n)$  deterministic update time algorithm for this problem.

None of the known algorithms for dynamically maintaining an MIS achieves an update time of  $o(n)$  in *dense graphs*. A recent result of Du and Zhang [7] partially addresses this lack of progress: they presented an “imperfect reduction” (to quote the authors in [7]) from the *Online Boolean Matrix-Vector Multiplication* problem to prove a conditional hardness result for the dynamic MIS problem (see, e.g., [9] for the role of this problem in proving conditional hardness result for dynamic problems).

This result hints that the update time of  $m^{1/2-\epsilon}$  or  $n^{1-\epsilon}$ , for any constant  $\epsilon > 0$ , may be a natural barrier for a large class of deterministic and randomized algorithms for dynamic MIS that satisfy a certain natural property (see [7] for the exact definition of this property and more details).

This state-of-affairs, namely, the lack of progress on obtaining the update time of  $o(n)$  for dynamic MIS in general on one hand, and the partial hardness result hinting that (essentially)  $\Omega(n)$  update time might be a natural barrier for this problem for a large class (but not all) of algorithms on the other hand, raises the following fundamental question:

QUESTION 1. *Can one maintain a maximal independent set in a dynamically changing graph with update time that is always  $o(n)$ ?*

**1.1 Our contribution** Our main result is a *positive* resolution of Question 1 in a strong sense:

THEOREM 1.1. *Starting from an empty graph on  $n$  fixed vertices, an MIS can be maintained over any sequence of edge insertions and deletions in  $\tilde{O}(\min\{m^{1/3}, \sqrt{n}\})$  amortized update time, where  $m$  denotes the dynamic number of edges, and the update time bound holds both in expectation and with high probability.<sup>2</sup>*

The proof of Theorem 1.1 is carried out in three stages. In the first stage, we provide a simple randomized algorithm for maintaining an MIS with update time  $\tilde{O}(n^{2/3})$ ; although simple, this algorithm already resolves Question 1. In the second stage, we generalize this simple algorithm to obtain an update time of  $\tilde{O}(m^{1/3})$ . Achieving the  $\tilde{O}(\sqrt{n})$  bound is more intricate; we reach this goal by carefully building on the ideas from the  $\tilde{O}(n^{2/3})$  and  $\tilde{O}(m^{1/3})$ -time algorithms.

Finding a maximal independent set is one of the most studied problems in distributed computing. It is thus important to provide an efficient distributed implementation of the proposed sequential dynamic algorithms. While the underlying distributed network is subject to topological updates (particularly edge updates) as in the sequential setting, the goal in the distributed setting is quite different: optimizing the (amortized) *round complexity*, *adjustment complexity* and *message complexity* of the distributed algorithm (see, e.g., [3, 5] for definitions). Achieving low amortized round and adjustment complexities is typically rather simple, and so the goal is to devise a

<sup>2</sup>We remark that the high probability guarantee holds when the number of updates is sufficiently large; see the formal statement of the results in later sections.

distributed algorithm whose amortized message complexity matches the update time of the proposed sequential algorithm. This goal was achieved by [3] and [8]. Similar to [3,8], our sequential algorithm can also be distributed, achieving an *expected* amortized message complexity of  $\tilde{O}(\min\{m^{1/3}, \sqrt{n}\})$ , in addition to  $O(1)$  amortized round and adjustment complexities, per each update. We omit the details of the distributed implementation of our algorithm as it follows in a straightforward way from our sequential algorithm using the ideas in [3].

**1.2 Overview of Our Techniques** At a very high level, the core idea behind all previous algorithms of [3,7,8] is the following. Partition the graph into “high” and “low” degree vertices (where the degree threshold changes from one algorithm to another); after each update, give *higher priority* to low degree vertices by adding them to the MIS even if they already have high degree neighbors in the MIS, then decide which of the high degree vertices need to leave the MIS as a result of adding these low degree vertices. This approach builds on the observations that all the low degree vertices can be handled efficiently using a simple algorithm with update time proportional to the maximum degree (see Lemma 2.1) and that the number of high degree vertices is small so they can be handled efficiently in a naive way (see Section 2 of [8] for an excellent overview of this approach). This approach however seems to be crucially stuck at an update time of  $\Theta(\sqrt{m})$  (which can be  $\Omega(n)$  for dense graphs), as further demonstrated by the (partial) conditional hardness result of [7].

In this paper we take a completely different approach to the problem that heavily relies on randomization. Instead of giving priority to low degree vertices, we give priority to a randomly selected set of (potentially very high) degree vertices, and we show that this simple twist guarantees that after removing these vertices and some of their neighbors, the degree of the remaining vertices is low. We then give a *higher priority* to the randomly chosen set of vertices, by making sure that the MIS in the remainder of the graph never interferes with the MIS induced by the random vertex set. In a sense, our approach can be viewed as a dual to the previous approaches. We now describe our approach in more detail.

**$\tilde{O}(n^{2/3})$ -Expected Amortized Update Time.** Our algorithm runs in multiple *phases*. At the beginning of each phase, we sample a set of  $\tilde{O}(n^{1/3})$  vertices uniformly at random among all vertices and compute an MIS  $\mathcal{M}_H$  for the subgraph induced on these vertices. These vertices form the high priority set and we never modify  $\mathcal{M}_H$  during

this phase. That is, if an update to  $\mathcal{M}_H$  is inevitable, which occurs only when the adversary inserts an edge between two vertices in  $\mathcal{M}_H$ , we terminate this phase and start the next one. We then use  $\mathcal{M}_H$  to partition the set of vertices in  $V \setminus H$  into two sets  $I_t$  and  $L_t$  for the graphs  $G_t(V, E_t)$  during this phase (where  $t$  is the current time step within the phase):  $I_t$  contains vertices incident to  $\mathcal{M}_H$  in  $G_t$  and  $L_t$  contains the remaining vertices. By maintaining an MIS  $\mathcal{M}_{L_t}$  of the subgraph induced on vertices in  $L_t$ , we obtain an MIS  $\mathcal{M} := \mathcal{M}_H \cup \mathcal{M}_{L_t}$  of the entire graph. Our task hence is to maintain an MIS of the subgraph induced on vertices in  $L_t$ , i.e., the “low priority” vertices. We remark that the set  $L_t$  may change over time, due to edge updates between vertices in  $L_t \cup I_t$  and vertices in  $H$ .

In order to maintain the MIS  $\mathcal{M}_{L_t}$ , we use a key property of the set  $L_t$  (and  $H$ ): the maximum degree in the subgraph induced on vertices in  $L_t$  is at most  $\tilde{O}(n^{2/3})$  with high probability. To see why this is true, note that if a vertex  $v$  has a larger degree in this subgraph, one of its neighbors there should have been sampled in  $H$  and joined  $\mathcal{M}_H$ , hence forcing  $v$  to be in  $I_t$  not  $L_t$ . This intuition is of course very informal. Formalizing it is nontrivial and involves combining the sample-and-prune technique of [15] for the MIS problem (similar to [13]) with additional ideas related to dynamic graphs. (See Lemma 2.2 and Lemma 3.1 for more details.) This allows us to maintain  $\mathcal{M}_{L_t}$  with  $\tilde{O}(n^{2/3})$  update time during this phase using an algorithm with update time proportional to the maximum degree (see Lemma 2.1). However, we should also take into account the time needed to “restart” a phase, which can be as large as  $O(n^2)$ . Performing the restart often would be troublesome. Fortunately, however, since set  $H$  is chosen uniformly at random from  $G$  and the adversary is oblivious to the randomness of the algorithm, the adversary has to use  $\Omega(n^{4/3})$  updates in expectation before it can force the algorithm to restart the phase. (We note that in the actual algorithm, we need to introduce multiple other criteria for terminating the phase to make the arguments work, but this upper bound continues to hold even with these additional termination criteria.) Consequently, we can relate the expected cost of all restarts to the number of updates and obtain an  $\tilde{O}(n^{2/3})$  expected amortized update algorithm. Proving that the same bound holds with high probability requires more efforts.

**$\tilde{O}(m^{1/3})$ -Expected Amortized Update Time.** Our algorithm in this part is a natural generalization of our  $\tilde{O}(n^{2/3})$  update time algorithm for all values of  $m$ . A key difference here is that we

also need to bound the cost of updates to the set  $L_t$ , i.e., the time needed to update the underlying graph of  $L_t$  when a vertex moves between  $I_t$  and  $L_t$ . This part of the algorithm combines new ideas with ideas from our previous approach [3] of maintaining local information of high degree vertices in  $I_t \cup L_t$  at all times in order to boost the performance of the algorithm when vertices are moved between these two sets.

**$\tilde{O}(n^{1/2})$ -Expected Amortized Update Time.** This brings us to our most technically involved algorithm. The general idea is to perform the partitioning of the graph  $G$  in the  $\tilde{O}(n^{2/3})$  update algorithm in a recursive way. In particular, we now have a *nested* sequence of phases as follows. In a level-1 phase, we sample a set  $H^1$  of vertices (with a different size than our previous algorithms), compute the MIS  $\mathcal{M}_{H^1}$ , and partition the vertices into  $I_t^1$  and  $L_t^1$  using  $\mathcal{M}_{H^1}$ . Now consider the graph  $G^1$  induced on vertices of  $L_t^1$ . The level-2 phase recursively applies the previous approach to this graph  $G^1$ , creating MIS  $\mathcal{M}_{H^2}$  and sets  $I_t^2$  and  $L_t^2$  of vertices inside  $L_t^1$ . We then define the graph  $G^2$  on  $L_t^2$  and continue to a level-3 phase and so on for  $R = \Theta(\log \log n)$  levels. As before, vertices in  $H^1, \dots, H^R$  have “high priority” in that we never modify  $\mathcal{M}_{H^r}$ , for  $r \in [R]$ , unless it is inevitable (more on this below). In the end, we obtain a collection of nested graphs  $G^1, G^2, \dots, G^R$ , along with multiple MISes  $\mathcal{M}_{H^1}, \dots, \mathcal{M}_{H^R}$ . We also maintain an MIS of the graph  $G^R$  explicitly, using the algorithm with max degree update time (i.e., Lemma 2.1). We ensure that the maximum degree of vertices in graphs  $G^1, \dots, G^R$  is likely decreasing (while the sizes of  $H^1, \dots, H^R$  are increasing), with vertices in  $G^R$  only having  $\tilde{O}(\sqrt{n})$  degree with high probability. Hence, during the most inner phase, i.e., a level- $R$  phase, we only need  $\tilde{O}(\sqrt{n})$  time to maintain the MIS.

The most challenging part in this algorithm is however to perform the “restarts” for different level phases when inevitable. When the adversary manages to insert an edge between two vertices in  $\mathcal{M}_{H^r}$  (for some  $r \in [R]$ ), we need to restart the computation of this phase (i.e., compute a new set  $H^r$  inside  $L_t^{r-1}$  and so on), as well as all inner phases, i.e., level- $(r+1)$  to level- $R$  phases (as they are defined on graph  $G^r$ , which may have completely changed). Note that all this is happening inside the graph  $G^{r-1}$ , and hence by bounding the max-degree of  $G^r$ , we can bound the time needed for this restart operation. We further use the fact that the adversary is oblivious to the randomness of the

algorithm to lower bound the expected number of updates needed for the adversary to force a level- $r$  phase restart, and show that the amortized cost of restarts is also  $\tilde{O}(\sqrt{n})$  in expectation. We note that a major difference in the cause of restarts in this new algorithm. Compared to the previous algorithms, the underlying graphs  $G^1, \dots, G^{R-1}$ , for which we recursively solve the problem, undergo not only *edge* insertions and deletions, but also *vertex* insertions and deletions. This follows from the definition of sets  $L^r$  and  $I^r$ . Handling this part requires multiple additional ideas compared to our  $\tilde{O}(n^{2/3})$  algorithm. They are discussed in Section 5.

## 2 Preliminaries

**Notation.** For a graph  $G(V, E)$ ,  $n$  denotes the number of vertices in  $V(G) := V$  and  $m$  denotes the number of edges in  $E(G) := E$ . For a set  $S \subseteq V$ , we define  $G[S]$  as the *induced* subgraph of  $G$  on vertices in  $S$ . We further define  $N_G(S)$  to be the set of vertices that are neighbor to at least one vertex in  $S$  in  $G$  (we may drop the subscript  $G$  when it is clear from the context). For a vertex  $v \in V$ , we define  $d_G(v)$  as the degree of  $v$  in  $G$ . Finally,  $\Delta(G)$  denotes the maximum degree in  $G$ .

**Greedy MIS.** The maximal independent set problem admits a simple greedy algorithm. Let  $G(V, E)$  be a graph and  $\pi := \pi(V)$  be any ordering of vertices in  $G$ .  $\text{Greedy}(G, \pi)$  iterates over vertices in  $G$  according to the ordering  $\pi$  and adds each vertex to the MIS if and only if none of its neighbors have already been chosen. It is immediate to verify that this algorithm indeed computes an MIS of  $G$  for any ordering  $\pi$ . Throughout this paper, we always assume that  $\pi$  is the *lexicographically-first ordering* of vertices and hence simply write  $\text{Greedy}(G)$  instead of  $\text{Greedy}(G, \pi)$ .

**2.1 A Deterministic  $O(\Delta)$ -Update Time Algorithm** We use the following simple algorithm for maintaining an MIS deterministically: every vertex maintains a *counter* of its neighbors in the MIS, and after any update to the graph, decides whether it should join or leave the MIS based on this information. Moreover, any vertex that joins or leaves the MIS uses  $O(\Delta)$  time to update the counter of its neighbors. While the worst case update time of this algorithm can be as large as  $\Omega(\Delta^2)$ , one can easily prove that *on average*, only  $O(\Delta)$  time is needed to process each update. This was first shown in [3] and further strengthened in [8].

LEMMA 2.1. ([3, 8]) *Starting from any graph  $G(V, E)$ , a maximal independent set can be maintained deterministically over any sequence of*

$K$  vertex or edge insertions and deletions in  $O(m + K \cdot \Delta)$  time, where  $\Delta$  is a fixed bound on the maximum degree in the graph and  $m = |E|$  is the original number of edges in  $G$ .

**2.2 Sample-and-Prune Technique for Computing an MIS** We also use a simple application of the *sample-and-prune* technique of [15] (see also [16]) originally introduced in context of streaming and MapReduce algorithms. To our knowledge, the following lemma was first proved in [13] following an approach in [1]. Intuitively speaking, it asserts that if we sample each vertex of the graph with probability  $p$ , compute an MIS of the sampled graph, and remove all vertices that are incident to this MIS, the degree of remaining vertices would be  $O(p^{-1} \cdot \log n)$ . For completeness, we present a self-contained proof of this lemma here (we note that our formulation is different from that of [13] and is tailored to our application).

**LEMMA 2.2.** (CF. [1, 13]) *Fix any  $n$ -vertex graph  $G(V, E)$  and a parameter  $p \in [0, 1)$ . Let  $S$  be a collection of vertices chosen by picking each vertex in  $V$  independently and with probability  $p$ . Suppose  $\mathcal{M} := \text{Greedy}(G[S])$  and  $U := V \setminus (\mathcal{M} \cup N_G(\mathcal{M}))$ . With probability  $1 - 1/n^4$ ,*

$$\Delta(G[U]) \leq 5p^{-1} \cdot \ln n.$$

*Proof.* Define  $\tau := 5p^{-1} \cdot \ln n$  and fix any vertex  $u$  in the original graph  $G$ . We prove that with high probability either  $u \notin U$  or  $d_{G[U]}(u) \leq \tau$  and then take a union bound on all vertices to conclude the proof.

Recall that  $\text{Greedy}(G[S])$  iterates over vertices of  $V$  in a lexicographically-first order. It skips a vertex if it is adjacent to the already constructed independent set  $\mathcal{M}$ . Otherwise, it selects it with probability  $p$  for inclusion in  $\mathcal{M}$ . Let  $v_1, \dots, v_{d_G(u)}$  be the neighbors of  $u$  in  $G$  ordered accordingly. When processing a vertex  $v_i$ , if  $v_i$  is not adjacent to  $\mathcal{M}$  computed so far, the probability that we pick  $v_i$  to join  $\mathcal{M}$  is exactly  $p$ . As such, if we encounter at least  $\tau$  such vertices in this process, the probability that we do not pick any of them is at most:

$$\begin{aligned} & \prod_{i=j}^{\tau} \Pr(v_{i_j} \text{ is not chosen} \mid v_{i_j} \text{ is not incident to the MIS}) \\ &= (1-p)^\tau \leq \exp(p \cdot 5p^{-1} \cdot \ln n) = \frac{1}{n^5}. \end{aligned}$$

Hence, we either did not encounter  $\tau$  vertices not adjacent to  $\mathcal{M}$ , which implies that  $d_{G[U]}(u) \leq \tau$ , or we did, which implies that with probability  $1 - 1/n^5$ ,

$u$  itself is neighbor to some vertex in  $\mathcal{M}$  (as by the calculation above, we would pick one of those at least  $\tau$  vertices) and hence does not belong to  $U$ . Taking the union bound over all vertices finalizes the proof. ■

### 3 An $\tilde{O}(n^{2/3})$ -Update Time Algorithm

We start with a simpler version of our algorithm and then build on this algorithm in Sections 4 and 5 to prove our Theorem 1.1.

**THEOREM 3.1.** *Starting from an empty graph on  $n$  vertices, a maximal independent set can be maintained via a randomized algorithm over any sequence of  $K$  edge insertions and deletions in  $O(K \cdot (n \cdot \log n)^{2/3})$  time in expectation and  $O(K \cdot (n \cdot \log n)^{2/3} + (n \log n)^{4/3})$  time with high probability.*

As a direct corollary of Theorem 3.1, we obtain the following result.

**COROLLARY 3.1.** *Starting from an empty graph on  $n$  vertices, a maximal independent set can be maintained via a randomized algorithm with  $O((n \log n)^{2/3})$  amortized update time with high probability, assuming the number of updates is sufficiently large, i.e., is  $\Omega((n \log n)^{2/3})$ .*

The algorithm in Theorem 3.1 works in *phases*. Each phase starts with a *preprocessing step* in which we initiate the data structure for the algorithm and in particular compute a partial MIS of the underlying graph with some useful properties. Next, during each phase, we have the *update step* which processes the updates to the graph until a certain condition is met, upon which we terminate this phase and start the next one. We now introduce our algorithm during one phase.

**The Preprocessing Step** The goal in this step is to find a partial MIS of the current graph with the following (informal) properties: (i) it should be “hard” for a *non-adaptive oblivious adversary* to update edges incident on this independent set, and (ii) maintaining an MIS in the remainder of the graph, i.e., after excluding these vertices and their neighbors from consideration, should be distinctly “easier”.

In the following, we prove that the sample-and-prune technique introduced in Section 2 can be used to achieve this task (we will pick an exact value for  $p$  below later but approximately  $p \approx n^{-2/3}$ ):

PreProcess( $G, p$ ):

1. Let  $H$  be a set chosen by picking each vertex in  $V(G)$  with probability  $p$  independently.
2. Compute  $\mathcal{M}_H := \text{Greedy}(G[H])$ .
3. Return  $(H, \mathcal{M}_H)$ .

Throughout this section, we use  $t_{\text{start}}$  to denote the time step in which PreProcess( $G, p$ ) is computed (hence  $G = G_{t_{\text{start}}}$ ). We define a partitioning of the vertices of  $G_t$  at any time  $t \geq t_{\text{start}}$ :

- $H$ : the set of vertices computed by PreProcess( $G_{t_{\text{start}}}, p$ ) (and not  $G_t$ ).
- $I_t := N_{G_t}(\mathcal{M}_H) \setminus H$ : the set of vertices incident on  $\mathcal{M}_H$  in the graph  $G_t$  that are not in  $H$ .
- $L_t := V \setminus (H \cup I_t)$ : the set of vertices is *neither* in  $H$  *nor* incident to  $\mathcal{M}_H$  in the graph  $G_t$ .

It is easy to see that in any time  $t \geq t_{\text{start}}$ ,  $(H, I_t, L_t)$  partitions the vertices of the graph. We emphasize that definition of  $H$  is with respect to the time step  $t_{\text{start}}$  and graph  $G_{t_{\text{start}}}$ , while  $I_t$  and  $L_t$  are defined for the graph  $G_t$  for  $t \geq t_{\text{start}}$ . This means that across time steps  $t \geq t_{\text{start}}$ , the set of vertices  $H$  is fixed but remaining vertices may move between  $I_t$  and  $L_t$ . We use this partitioning to define the following key time steps in the execution of the algorithm:

- $t_H \geq t_{\text{start}}$ : the first time step  $t$  in which  $G_t[H] \neq G_{t_{\text{start}}}[H]$  (recall that  $H$  and  $\mathcal{M}_H$  were computed with respect to  $G_{t_{\text{start}}}$  and not  $G_t$ ).
- $t_I \geq t_{\text{start}}$ : the first time step  $t$  in which the total number of times (since  $t_{\text{start}}$ ) that vertices have moved from  $I_s$  to  $L_{s+1}$ , for  $s < t$ , reaches  $2p^{-1}$ .
- $t_L \geq t_{\text{start}}$ : the first time step  $t$  in which  $\Delta(G_t[L_t]) > 5p^{-1} \cdot \ln n$ .
- $t_{\text{end}} := \min\{t_H, t_I, t_L, t_{\text{start}} + T\}$  where  $T := \frac{1}{6p^2}$ : the time step in which we terminate this phase (in other words, if any of the conditions above happen, the phase finishes and the next phase starts).

By definition above, each phase starts at time step  $t_{\text{start}}$  and ends at time step  $t_{\text{end}}$  and has length at most  $T = \frac{1}{6p^2}$ . We say that a phase is *successful* iff  $t_{\text{end}} = t_{\text{start}} + T$ .

We stress that the notation above is defined with respect to an arbitrary *fixed* phase, starting at time

$t_{\text{start}}$ . Whenever any of these time steps happen we terminate this phase and move to the next one. As such, throughout this section, all of these time steps are defined with respect to the original set  $H$  and the time step  $t_{\text{start}}$  in which this phase has started.

In the following, we prove that every phase is successful with at least a constant probability (this fact will be used later to argue that the cost of preprocessing steps can be amortized over the large number of updates between them).

LEMMA 3.1. *Any given phase is successful, i.e., has  $t_{\text{end}} = t_{\text{start}} + T$ , with probability at least  $1/2$ .*

*Proof.* The lemma is proved in the following three claims which bound  $t_H$ ,  $t_I$ , and  $t_L$ , respectively. All claims crucially use the fact that the adversary is non-adaptive and oblivious and hence we can fix its updates beforehand.

CLAIM 3.2.  $\Pr(t_H < t_{\text{start}} + T) \leq \frac{1}{6}$ .

*Proof.* For any  $t \geq t_{\text{start}}$ , let  $e_t := (u_t, v_t)$  denote the edge updated by the adversary at time  $t$ . We consider the randomness in PreProcess( $G_{t_{\text{start}}}, p$ ). The probability that both  $u_t$  and  $v_t$  belong to  $H$  is exactly  $p^2$ . For any  $t \in [t_{\text{start}}, t_{\text{start}} + T)$ , define an indicator random variable  $X_t$  which is 1 iff  $(u_t, v_t)$  belongs to  $G[H]$ . Let  $X := \sum_t X_t$ . In order for  $G_t[H]$  to no longer be equal to  $G_{t_{\text{start}}}[H]$  for some  $t \in [t_{\text{start}}, t_{\text{start}} + T)$ , at least one of these  $T - 1$  updates needs to have both endpoints in  $H$ . As such,

$$\begin{aligned} \Pr(t_H < t_{\text{start}} + T) &\leq \Pr(X \geq 1) \leq \mathbb{E}[X] \\ &= (T - 1) \cdot p^2 \leq \frac{1}{6p^2} \cdot p^2 \\ &= \frac{1}{6}, \end{aligned}$$

where the second inequality is by Markov bound. ■

CLAIM 3.3.  $\Pr(t_I < t_{\text{start}} + T) \leq \frac{1}{6}$ .

*Proof.* For any  $t \geq t_{\text{start}}$ , let  $e_t := (u_t, v_t)$  denote the edge updated by the adversary at time  $t$ . By the randomness in PreProcess( $G_{t_{\text{start}}}, p$ ), the probability that at least one endpoint of  $e_t$  belongs to  $H$  is  $2p - p^2 \leq 2p$ . For any  $t \in [t_{\text{start}}, t_{\text{start}} + T)$ , define an indicator random variable  $Y_t$  which is 1 iff at least one of  $u_t$  or  $v_t$  belong to  $H$ . Let  $Y := \sum_t Y_t$ .

The only way a vertex from  $I$  moves to  $L$  is that an edge incident on this vertex with other endpoint in  $\mathcal{M}_H$  is deleted (and this vertex has no other edge to  $\mathcal{M}_H$  either). For this to happen  $2p^{-1}$  times (as in definition of  $t_I$ ), we need to have at least  $2p^{-1}$

updates in the range  $[t_{\text{start}}, t_{\text{start}} + T)$  with at least one endpoint in  $H$  (recall that  $\mathcal{M}_H \subseteq H$ ). As such,

$$\begin{aligned} \Pr(t_I < t_{\text{start}} + T) &\leq \Pr(Y \geq 2p^{-1}) \leq \mathbb{E}[Y] \cdot \frac{p}{2} \\ &\leq (T-1) \cdot 2p \cdot \frac{p}{2} \leq \frac{1}{6p^2} \cdot p^2 \\ &= \frac{1}{6}, \end{aligned}$$

where the second inequality is by Markov bound.  $\blacksquare$

CLAIM 3.4.  $\Pr(t_L < t_{\text{start}} + T) \leq \frac{1}{n^2}$ .

*Proof.* Fix a time step  $t \in [t_{\text{start}}, t_{\text{start}} + T)$ . Let  $G_t$  be the graph at this time step and for the sake of analysis define another graph  $G_t^H$  where  $G_t^H[H] = G_{t_{\text{start}}}^H[H]$  and  $G_t^H$  is equal to  $G_t$  everywhere else (i.e., we replaced the induced subgraph of  $G_t$  on vertices in  $H$  with the corresponding part in  $G_{t_{\text{start}}}$ ). Since  $G_{t_{\text{start}}}^H[H] = G_t^H[H]$  and  $\mathcal{M}_H = \text{Greedy}(G_{t_{\text{start}}}^H[H])$ , we also have  $\mathcal{M}_H = \text{Greedy}(G_t^H[H])$ . Thus, we can apply Lemma 2.2 with  $S = H$  and  $U = L_t$  (recall that  $H$  and  $L_t$  are defined w.r.t to the same fixed  $H$  computed at time  $t_{\text{start}}$ ) and conclude that with probability  $1 - 1/n^4$ , we have that  $\Delta(G_t^H[L_t]) \leq 5p^{-1} \ln n$ . But since  $G_t^H[L_t] = G_t[L_t]$ , it follows that  $\Delta(G_t[L_t]) \leq 5p^{-1} \ln n$  as well. Taking a union bound over these  $\leq n^2$  graphs finalizes the proof.  $\blacksquare$

By applying union bound to Claims 3.2, 3.3, and 3.4, the probability that  $t_{\text{end}} = \min\{t_H, t_I, t_L\} < t_{\text{start}} + T$  is at most  $1/6 + 1/6 + 1/n^2 \leq 1/2$ , finalizing the proof. Lemma 3.1.  $\blacksquare$

We conclude this section with the following straightforward lemma.

LEMMA 3.5.  $\text{PreProcess}(G, p)$  takes  $O(m + n)$  time where  $m := |E(G)|$ .

**The Update Algorithm** As argued before, each phase continues between time steps  $t_{\text{start}}$  and  $t_{\text{end}}$  where the latter is smaller than or equal to time steps  $t_H, t_I$  and  $t_L$ . We have the following invariant.

INVARIANT 1. At any time step  $t \in [t_{\text{start}}, t_{\text{end}})$  inside one phase:

(i)  $\mathcal{M}_H$  is an MIS of the graph  $G_t[H]$ ,

(ii)  $\Delta(G_t[L_t]) = O(p^{-1} \cdot \log n)$ .

Moreover, throughout the phase, at most  $O(p^{-1})$  vertices are moved from  $I$  to  $L$ .

We note that the first property above is simply because  $G_t[H] = G_{t_{\text{start}}}^H[H]$  as  $t < t_H$  and hence  $\mathcal{M}_H$

is also an MIS of  $G_t[H]$ . The second property is by definition of  $t_L$  and the last one is by definition of  $t_I$ .

Our update algorithm simply maintains the graph  $G_t[L_t]$  at all time and run the basic deterministic algorithm in Lemma 2.1 on  $G_t[L_t]$  to maintain an MIS  $\mathcal{M}_{L_t}$  of  $G_t[L_t]$ . The full MIS maintained by the dynamic algorithm is then  $\mathcal{M}_H \cup \mathcal{M}_{L_t}$ .

We now describe the update algorithm in more details. For any vertex  $v \in V$ , we maintain whether it currently belongs to  $H$ ,  $I_t$ , or  $L_t$ . Additionally, for any vertex in  $I_t \cup H$ , we maintain a list of its neighbors in  $\mathcal{M}_H$ . Finally, we also maintain the graph  $G_t[L_t]$ , which involves storing, for each vertex  $v \in L_t$ , the set of all of its neighbors in  $L_t$ . Note that both edges and vertices (as opposed to only edges) may be inserted to or deleted from  $G_t[L_t]$  by the algorithm (and as such, we crucially use the fact that the algorithm in Lemma 2.1 can process vertex updates as well). Fix a time  $t \in [t_{\text{start}}, t_{\text{end}}]$  and let  $e_t = (u_t, v_t)$  be the updated edge. We consider the following cases:

- **Case 1.** Updates that cannot impact the partitioning  $(H, I_t, L_t)$  of vertices:

- **Case 1-a.** Both  $u_t$  and  $v_t$  belong to  $H$ . This update means that  $t = t_H$  as the graph  $G_t[H]$  is updated and hence this update concludes this phase (and is processed in the next phase).

- **Case 1-b.** Both  $u_t$  and  $v_t$  belong to  $I_{t-1}$ . There is nothing to do in this case.

- **Case 1-c.** Both  $u_t$  and  $v_t$  belong to  $L_{t-1}$ . We need to update the edge  $(u_t, v_t)$  in the graph  $G_t[L_t]$  and pass this edge update to the algorithm in Lemma 2.1 on  $G_t[L_t]$ .

- **Case 1-d.**  $u_t$  belongs to  $I_{t-1}$  and  $v_t$  belongs to  $L_{t-1}$  (or vice versa). There is nothing to do in this case.

- **Case 2.** Updates that can (potentially) change the partitioning  $(H, I_t, L_t)$  of vertices:

- **Case 2-a.**  $u_t$  is in  $H$  and  $v_t$  is in  $I_{t-1}$  (or vice versa). If  $e_t$  is inserted, the partitioning  $(H, I_t, L_t)$  remains the same and there is nothing to do except for updating the list of neighbors of  $v_t$  in  $\mathcal{M}_H$ . However, if  $e_t$  is deleted, it might be that  $v_t$  needs to be removed from  $I_t$  and inserted to  $L_t$  instead (if it is no longer incident on  $\mathcal{M}_H$ ). If so, we iterate over all neighbors of  $v_t$  and find the ones which are in  $L_t$ . We then insert  $v_t$  with all its incident edges to  $G_t[L_t]$  and pass this vertex update to the algorithm in Lemma 2.1 on  $G_t[L_t]$ .

- **Case 2-b.**  $u_t$  is in  $H$  and  $v_t$  is in  $L_{t-1}$  (or vice versa). If  $e_t$  is deleted, the partitioning  $(H, I_t, L_t)$  remains the same and there is nothing to do. However, if  $e_t$  is inserted, it might be that  $v_t$  needs to leave  $L_t$  and join  $I_t$  (if  $u_t$  belongs to  $\mathcal{M}_H$ ). If so, we delete  $v_t$  with all its incident edges in  $L_t$  from  $G_t[L_t]$  and run the algorithm in Lemma 2.1 to process this vertex update in  $G_t[L_t]$ .

The cases above cover all possible updates. By the correctness of the deterministic algorithm in Lemma 2.1,  $\mathcal{M}_{L_t}$  is a valid MIS of  $G_t[L_t]$ . Since all vertices in  $I_t$  are incident to some vertex in  $\mathcal{M}_H$ , it is immediate to verify that  $\mathcal{M}_H \cup \mathcal{M}_{L_t}$  is an MIS of the graph  $G$  for any time step  $t \in [t_{\text{start}}, t_{\text{end}}]$  by Invariant 1. It only remains to analyze the running time of the update algorithm.

**LEMMA 3.6.** *Let  $K$  denote the number of updates in a particular phase. The update algorithm maintains an MIS of the graph in this phase in  $O(n^2 + p^{-1} \cdot n + K \cdot p^{-1} \cdot \log n)$  time.*

*Proof.* The cost of bookkeeping the data structures in the update algorithm is  $O(1)$  per each update. The two main time consuming steps are hence maintaining an MIS in the graph  $G_t[L_t]$  and maintaining the graph  $G_t[L_t]$  itself.

The former task, by Lemma 2.1, requires  $O(n^2 + K \cdot \Delta^*)$  time in total where  $\Delta^* := \max_t \Delta(G_t[L_t])$ , which by Invariant 1 is  $O(p^{-1} \log n)$ . Hence, this part takes  $O(n^2 + K \cdot p^{-1} \log n)$  time in total.

For the latter task, performing edge updates (in Case 1-c) can be done with  $O(1)$  time per each update. Making vertex-deletion updates (in Case 2-b) can also be done in  $O(\Delta^*)$  time per update as we only need to iterate over neighbors of the updated vertex in  $G_t[L_t]$ . However, performing the vertex-insertion updates (in Case 2-a) requires iterating over all neighbors of the inserted vertex (in  $G_t$  not only  $G_t[L_t]$ ) and hence takes  $O(n)$  time. Nevertheless, by Invariant 1, the total number of such vertex updates is  $O(p^{-1})$  and hence their total running time is  $O(p^{-1} \cdot n)$ . ■

**Proof of Theorem 3.1** We are now ready to prove Theorem 3.1. The correctness of the algorithm immediately follows the discussion after Invariant 1 as  $\mathcal{M}_H \cup \mathcal{M}_{L_t}$  is an MIS of the graph  $G_t$ . Hence, it only remains to bound the amortized update time of the algorithm.

Fix a sequence of  $K$  updates, and let  $P_1, \dots, P_k$  denote the different phases of the algorithm over this sequence (i.e., each  $P_i$  corresponds to the updates

inside one phase). The time spent by the overall algorithm in each phase  $i \in [k]$  is  $O(n^2)$  in the preprocessing step (by Lemma 3.5), and  $O(n^2 + p^{-1} \cdot n + |P_i| \cdot p^{-1} \cdot \log n)$  (by Lemma 3.6). As such, the total running time is

$$(3.1) \quad O(k \cdot (n^2 + p^{-1}n) + K \cdot p^{-1} \cdot \log n),$$

since  $\sum_i |P_i| = K$ . So to finalize the proof, we only need to bound the number of phases, which is done in the following two lemmas.

**LEMMA 3.7.**  $\mathbb{E}[k] = O(K \cdot p^2)$ .

*Proof.* Recall that a phase  $P_i$  is called successful iff  $|P_i| = T (= \frac{1}{6p^2})$ . The probability that any phase  $P_i$  is successful is at least  $1/2$  by Lemma 3.1, regardless of what has happened in the previous phases (unless there are no updates left in which case this is going to be the last phase).

Notice that any successful phase includes  $T$  updates and hence we can have at most  $K/T$  long phases (even if we assume short phases include no updates). Consider the following randomized process: we have a coin which has at least  $1/2$  chance of tossing head in each trial regardless of the outcome in previous tosses; how many times in expectation do we need to toss this coin to see  $K/T$  heads? It is immediate to verify that  $\mathbb{E}[k]$  is at most this number (plus one) and this process stochastically dominates a binomial process with probability  $1/2$  in each trial. As such, the expected number of coin tosses in this process is  $2K/T$ . Hence  $\mathbb{E}[k] \leq 2K/T + 1 = O(K \cdot p^2)$ . ■

By plugging in Lemma 3.7 in Eq (3.1), the expected running time of the algorithm is  $O(K \cdot (p^2 \cdot n^2 + p \cdot n) + K \cdot p^{-1} \log n)$ . By picking  $p := \frac{(\log n)^{1/3}}{n^{2/3}}$ , the expected running time of the algorithm would be  $O(K \cdot (n \cdot \log n)^{2/3})$  time, proving the bound on expected amortized update time in Theorem 3.1.

We now prove the high probability bound on the running time.

**LEMMA 3.8.** *With probability  $1 - \exp(-K \cdot p^2/10)$ ,  $k = O(K \cdot p^2)$ .*

*Proof.* Recall the coin tossing process described in the proof of Lemma 3.7. Consider the event that among the first  $4K/T$  coin tosses, there are at most  $K/T$  heads. The probability of this event is at most  $\exp(-K/10T)$  by a simple application of Chernoff bound. On the other hand, the probability of this event is at least equal to the probability that among the first  $4K/T$  phases of the algorithm, there are at most  $K/T$  long phases. This concludes the proof



of first part as we cannot have more than  $K/T$  long phases among  $K$  updates (each long phase “consumes”  $T$  updates). ■

By the choice of  $p = \frac{(\log n)^{1/3}}{n^{2/3}}$ , if  $K \geq 10n^{4/3} \cdot \log^{1/3} n$ , then by Lemma 3.8 and Eq (3.1), the running time of the algorithm is  $O(K \cdot (n \cdot \log n)^{2/3})$  with high probability, finalizing the proof of this part also. If however  $K < 10n^{4/3} \log^{1/3} n$ , we only need to have  $2 \log n$  successful phases to process all the updates. In this case, since every phase is successful with constant probability, with high probability we only need to consider  $O(\log n)$  phases before we are done. Moreover, note that when the number of updates is at most  $O(n^{4/3} \log^{1/3} n)$ , the total number of edges in the graph is also  $O(n^{4/3} \log^{1/3} n)$  only and the preprocessing time takes  $O(n^{4/3} \log^{1/3} n)$  per each phase as opposed to  $O(n^2)$ . This means that the total running time in this case is with high probability at most  $O(n^{4/3} \cdot \log^{4/3} n)$  (for preprocessing) plus  $O(K \cdot (n \log n)^{2/3})$  (time spent inside the phases). This concludes the proof of Theorem 3.1.

#### 4 An Improved $\tilde{O}(m^{1/3})$ -Update Time Algorithm

We now show that one can alter the algorithm in Theorem 3.1 to obtain improved performance for sparser graphs with  $m = o(n^2)$  edges. Formally,

**THEOREM 4.1.** *Starting from an empty graph, a maximal independent set can be maintained via a randomized algorithm over any sequence of edge insertions and deletions in  $O(m^{1/3} \log m)$  amortized update time both in expectation and with high probability, where  $m$  is the dynamic number of edges.*

Note that the bound in Theorem 4.1 is never worse than the one in Theorem 3.1 (modulo low-order terms which we did not optimize in Theorem 4.1), and improves upon it whenever  $m = o(n^2)$ . The following lemma is a somewhat weaker looking version of Theorem 4.1. However, we prove next that this lemma is all we need to prove Theorem 4.1.

**LEMMA 4.1.** *Starting with any arbitrary graph on  $m \geq 100$  edges, a maximal independent set can be maintained via a randomized algorithm over any sequence of  $K = \Omega(m)$  edge insertions and deletions in  $O(K \cdot m^{1/3} \log m)$  time in expectation and with high probability, as long as the number of edges in the graph remains within a factor 2 of  $m$ .*

We first prove that this lemma implies Theorem 4.1. The proof of this part is standard (see, e.g. [3]) and is only provided for completeness.

*Proof.* [Proof of Theorem 4.1] When  $m \leq 100$ , naive re-computation of MIS after every update takes constant time and hence there is nothing to do. Hence, we only need to consider the case where  $m \geq 100$ . The idea is to run the algorithm in Lemma 4.1 until the number of edges deviate from  $m$  by a factor more than 2, upon which, we terminate the algorithm and restart the process. As the total number of updates is  $\Omega(m)$ , we can apply Lemma 4.1 and obtain a bound of  $O(m^{1/3} \log m)$  on the expected amortized update time. Moreover, we can “charge” the  $O(m)$  time needed to restart the process to the  $\Omega(m)$  updates happening in this phase and obtain the final bound (see Lemma 4.6). ■

The rest of this section is devoted to the proof of Lemma 4.1. The algorithm in Lemma 4.1 is similar to the one in Theorem 3.1 and in particular again executes multiple phases each starting by the same preprocessing step (although with change of parameters) followed by the update algorithm throughout the phase. We now describe the preprocessing step and the update algorithm inside each phase. Recall that  $m = m_{t_{\text{start}}}$  denotes the number of edges in the graph  $G_{t_{\text{start}}}$  and that throughout the proof of Lemma 4.1,  $m$  provides a 2-approximation to the number of edges in the underlying graph at any time step.

**The Preprocessing Step** Let  $t_{\text{start}}$  again denote the first time step in this phase. The preprocessing step of the new algorithm is as before where we run  $\text{PreProcess}(G_{t_{\text{start}}}, p)$  for  $p = m^{-1/3}$  (this value of  $p$  generalizes the one in Section 3 which was  $\approx n^{-2/3}$ ). We define the partitioning  $(H, I_t, L_t)$  of vertices as before. However, we change the stopping criteria of the phase and definition of time steps  $t_H, t_I, t_L$  (we emphasize that these choices are natural generalization of the bounds in the previous section):

- $t_H \geq t_{\text{start}}$ : the first time step  $t$  in which  $G_t[H] \neq G_{t_{\text{start}}}[H]$  (recall that  $H$  and  $\mathcal{M}_H$  were computed with respect to  $G_{t_{\text{start}}}$  and not  $G_t$ ).
- $t_I \geq t_{\text{start}}$ : the first time step  $t$  in which the total number of times (since  $t_{\text{start}}$ ) that vertices have moved from  $I_s$  to  $L_{s+1}$ , for  $s < t$ , reaches  $m^{1/3}$ .
- $t_L \geq t_{\text{start}}$ : the first time step  $t$  in which  $\Delta(G_t[L_t]) > 20m^{1/3} \cdot \ln(m)$ .
- $t_{\text{end}} := \min\{t_H, t_I, t_L, t_{\text{start}} + T\}$  where  $T := \frac{1}{6} \cdot m^{2/3}$ : the time step in which we terminate this phase.

We again say that a phase is *successful* if  $t_{\text{end}} = t_{\text{start}} + T$ , i.e., we process  $T$  updates in the phase

before terminating. Similar to Lemma 3.1, we prove that each phase is successful with at least a constant probability.

LEMMA 4.2. *Any given phase is successful with probability at least  $1/2$ .*

*Proof.* The proof is similar to Lemma 3.1 and is based on the fact that the adversary is non-adaptive and oblivious.

CLAIM 4.3.  $\Pr(t_H < t_{\text{start}} + T) \leq \frac{1}{6}$ .

*Proof.* The proof is identical to Claim 3.2 by substituting the new values of  $p$  and  $T$ . ■

CLAIM 4.4.  $\Pr(t_I < t_{\text{start}} + T) \leq \frac{1}{6}$ .

*Proof.* Again, the proof is identical to Claim 3.3 by substituting the new values of  $p$  and  $T$ . ■

CLAIM 4.5.  $\Pr(t_L < t_{\text{start}} + T) \leq \frac{1}{m^2}$ .

*Proof.* Fix the graphs  $G_t$  for  $t \in [t_{\text{start}}, t_{\text{start}} + T)$  and note that  $G_t$  has at most  $4m$  vertices with non-zero degree (as number of edges in  $G_t$  is at most  $2m$ ) and we can ignore vertices with degree zero as they will not affect the following calculation. The rest of the proof is as in Claim 3.4 as by Lemma 2.2, with choice of  $S = H$  and  $U_t = L_t$  for any graph  $G_t$  (with at most  $4m$  vertices), with probability  $1 - 1/m^4$ ,  $\Delta(G_t[L_t]) \leq 10p^{-1} \cdot \ln(4m) \leq 20m^{1/3} \cdot \ln m$ . Taking a union bound over these  $\leq m^2$  graphs finalizes the proof. ■

By applying union bound to Claims 4.3, 4.4, and 4.5, the probability that  $\min\{t_H, t_L, t_I\} < t_{\text{start}} + T$  is at most  $1/6 + 1/6 + 1/m^2 \leq 1/2$  (as  $m \geq 100$ ), finalizing the proof of Lemma 4.2. ■

By Lemma 3.5, the preprocessing step of this algorithm takes  $O(m + n)$  time. However, a simple trick can reduce the running time to only  $O(m)$  as follows.

LEMMA 4.6. *The preprocessing step of the new algorithm can be implemented in  $O(m)$  time.*

*Proof.* Initially, there are at most  $4m$  vertices in the preprocessing step that have non-zero degree (we keep track of non-isolated vertices throughout the whole algorithm). Hence, instead of picking the set  $H$  from all of  $V$ , we only pick it from the vertices with non-zero degree, which can be done in  $O(m)$  time. Later in the algorithm, whenever a new vertex is given an edge in this phase, we toss a coin and decide to add it to  $H$  with probability  $p$  which can be done in  $O(1)$  time. We then process this update as before as if this new vertex always belonged to  $H$ . It is immediate to verify that this does not change any part of the algorithm. ■

**The Update Algorithm** We now describe the new update algorithm. Firstly, similar to Invariant 1 in the previous section, we have that,

INVARIANT 2. *At any time step  $t \in [t_{\text{start}}, t_{\text{end}}]$  inside one phase:*

- (i)  $\mathcal{M}_H$  is an MIS of the graph  $G_t[H]$ ,
- (ii)  $\Delta(G_t[L_t]) = O(m^{1/3} \log(m))$ .

*Moreover, throughout the phase, at most  $O(m^{1/3})$  vertices are moved from  $I$  to  $L$ .*

The update algorithm is similar to the one in previous section: we maintain the graph  $G_t[L_t]$  and use the algorithm in Lemma 2.1 to maintain an MIS  $\mathcal{M}_{L_t}$  in  $G_t[L_t]$ . The only difference is in how we maintain the graph  $G_t[L_t]$ . In order to do this, we present a simple data structure.

**The Data Structure.** As before, we maintain the list of all neighbors of each vertex, as well as the set  $H$ ,  $I_t$ , or  $L_t$  that it belongs to for each vertex. Clearly, this information can be updated in  $O(1)$  time per each update. In addition to the partition  $(H, I_t, L_t)$ , we also partition vertices based on their degree in the original graph *at the beginning of the phase*, i.e., in  $G_{t_{\text{start}}}$ . Specifically, we define  $V_{\text{high}}$  to be the set of vertices with degree at least  $m^{2/3}$  in  $G_{t_{\text{start}}}$  and  $V_{\text{low}} := V \setminus V_{\text{high}}$  to be the remaining vertices. Note that this partitioning is defined with respect to the graph  $G_{t_{\text{start}}}$  and does *not* change throughout the phase. We have the following simple claim.

CLAIM 4.7. *Throughout one phase:*

1.  $|V_{\text{high}}| = O(m^{1/3})$ .
2. *For any vertex  $v \in V_{\text{low}}$  and any graph  $G_t$  for  $t \geq t_{\text{start}}$ , degree of  $v$  in  $G_t$  is  $O(m^{2/3})$ .*

*Proof.* The first is simply because each vertex in  $V_{\text{high}}$  has degree at least  $m^{2/3}$  and the total number of edges is at most  $2m$ . The second part is because the total number of updates inside a phase is at most  $\frac{1}{6} \cdot m^{2/3}$  by the definition of  $t_{\text{end}}$  and hence even if they are all incident on a vertex in  $V_{\text{low}}$ , the degree of the vertex is at most  $\frac{7}{6} \cdot m^{2/3}$ . ■

Finally, for any vertex  $v \in V_{\text{high}}$ , we maintain a list of all of its neighbors in  $L_t$  as follows: whenever a vertex moves between  $I_t$  and  $L_t$ , it iterates over all vertices in  $V_{\text{high}}$  and informs them of this update. This way, vertices in  $V_{\text{high}}$  are always aware of their neighborhood in  $L_t$ . The remaining vertices also have a relatively small degree and hence whenever needed,

we could simply iterate over all their neighbors and find the ones in  $L_t$ . As a result of this, we have the following invariant.

**INVARIANT 3.** *At any time step  $t \in [t_{\text{start}}, t_{\text{end}}]$  inside one phase after updating  $e_t = (u_t, v_t)$ :*

- (i) *We can find the list of all neighbors of  $u_t$  and  $v_t$  that belong to  $L_t$  in  $O(m^{2/3})$  time.*
- (ii) *Updating the data structure after the update takes  $O(m^{1/3})$  time.*

*Proof.* For vertices  $u_t$  and  $v_t$  in  $V_{\text{high}}$ , we have maintained the list of their neighbors explicitly and hence we can directly return this list. For vertices  $u_t$  and  $v_t$  in  $V_{\text{low}}$ , we can simply iterate over their  $O(m^{2/3})$  neighbors (by Claim 4.7) and check which one belongs to  $L_t$  and create the list in  $O(m^{2/3})$  time. Finally, the update time is  $O(m^{1/3})$  as there are only  $O(m^{1/3})$  vertices in  $V_{\text{high}}$  (by Claim 4.7) and each vertex is only updating these vertices per update. ■

**Processing Each Update.** We process each update as in the previous section, with the difference that we use Invariant 3, for maintaining the graph  $G_t[L_t]$ . To be more specific, in Case 2-a, where a vertex may be inserted in  $G_t[L_t]$ , we use the list in Invariant 3, to find all neighbors of this vertex in  $L_t$  and then pass this vertex-update to the algorithm of Lemma 2.1 on  $G_t[L_t]$ . The remaining cases are handled exactly as before.

The correctness of the algorithm follows as before and we only analyze the running time of the update algorithm.

**LEMMA 4.8.** *Fix any phase and let  $K$  denote the number of updates inside this phase. The update algorithm maintains an MIS of the input graph (deterministically) in  $O(m + K \cdot m^{1/3} \cdot \log m)$  time.*

*Proof.* By Invariant 3, updating the data structure takes  $O(K \cdot m^{1/3})$  time. Maintaining the MIS in the graph  $G_t[L_t]$  also requires  $O(m + K \cdot m^{1/3} \log m)$  time by Lemma 2.1 (as max-degree is  $O(m^{1/3} \log m)$  by Invariant 2). Finally, by Invariant 3, we can find the neighbors of any updated vertex in  $L_t$  in  $O(m^{2/3})$  time. Since, the total number of times we need to find these neighbors is  $O(m^{1/3})$  by Invariant 2 (as we only need this operation when a vertex moves from  $I$  to  $L$ ), the total time needed for this part is also  $O(m)$ , finalizing the proof. ■

**Proof of Lemma 4.1** The correctness of the algorithm immediately follows from Lemma 4.8, hence, it only remains to bound the amortized update time

of the algorithm. Fix a sequence of  $K$  updates, and let  $P_1, \dots, P_k$  denote the different phases of the algorithm over this sequence (i.e., each  $P_i$  corresponds to the updates inside one phase). The time spent by the overall algorithm in each phase  $i \in [k]$  is  $O(m)$  in the preprocessing step (by Lemma 4.6), and  $O(m + |P_i| \cdot m^{1/3} \log m)$  (by Lemma 4.8). As such, the total running time is  $O(k \cdot m + K \cdot m^{1/3} \log m)$  (since  $\sum_i |P_i| = K$ ). So to finalize the proof, we only need to bound the number of phases (proof is identical to Lemmas 3.7 and 3.8 by setting  $p = m^{-1/3}$ ).

**LEMMA 4.9.**  $\mathbb{E}[k] = O(K/m^{2/3})$ . *Moreover, w.p.  $1 - \exp(-K \cdot p^2/10)$ ,  $k = O(K \cdot m^{2/3})$ .*

By Lemma 4.9, the expected running time of the algorithm is  $O(K \cdot m^{1/3} + K \cdot m^{1/3} \ln m)$ , concluding the proof of expectation-bound in Lemma 4.1. The extension to the high probability result now is exactly the same as in Lemma 3.8, as  $K = \Omega(m) \gg m^{2/3} \log m$ . This concludes the proof of Lemma 4.1.

## 5 Main Algorithm: An $\tilde{O}(\sqrt{n})$ -Update Time Algorithm

We now present our main algorithm for maintaining an MIS in a dynamic graph with  $\tilde{O}(\sqrt{n})$  expected amortized update time.

**THEOREM 5.1.** *Starting from an empty graph on  $n$  vertices, a maximal independent set can be maintained via a randomized algorithm over any sequence of  $K$  edge insertions and deletions in  $O(K \cdot \sqrt{n} \cdot \log^2 n \cdot \log \log n)$  time in expectation and in  $O(K \cdot \sqrt{n} \cdot \log^2 n \cdot \log \log n + n^2 \log^3 n)$  time with high probability.*

A direct corollary of Theorem 5.1 is the following.

**COROLLARY 5.1.** *Starting from an empty graph on  $n$  vertices, a maximal independent set can be maintained via a randomized algorithm with  $\tilde{O}(\sqrt{n})$  amortized update time with high probability, assuming the number of updates is sufficiently large, i.e., is  $\Omega(n^{3/2} \cdot \log^3(n))$ .*

The improvement in Theorem 5.1 over our previous algorithm in Theorem 3.1 is obtained by using a *nested collection of phases* instead of just one phase. Let  $R := 2 \log \log n$ . We maintain  $R$  subgraphs of the input graph at any time step of the algorithm, referred to as *level graphs*. For any level  $r \in [R]$ , we compute and maintain the subgraph at level  $r$  in a *level- $r$  phase*. A phase as before consists of a preprocessing step, followed by update steps during the phase, and a termination criteria for the phase. Moreover, the phases across different levels are nested in a way that a level-1 phase consists of multiple level-2

phases, a level-2 phase contain multiple level-3 phases and so on. We now describe our algorithm in more detail starting with the nested family of level graphs.

**Level Graphs** Our approach is based on computing and maintaining a collection of graphs  $G_t^1, \dots, G_t^R$ , referred to as level graphs, which are subgraphs of  $G_t$  and a collection of independent sets  $\mathcal{M}_t^1, \dots, \mathcal{M}_t^R, \mathcal{M}_t^*$ . We maintain the following main invariant in our algorithm (we prove different parts of this invariant in this and the next two sections).

**INVARIANT 4. (Main Invariant)** *At any time step  $t$  and for any  $r \in [R]$ :*

1.  $\mathcal{M}_t^1 \cup \dots \cup \mathcal{M}_t^R \cup \mathcal{M}_t^*$  is a maximal independent set of  $G_t$ .
2.  $\Delta(G_t^r) \leq \Delta_r$  (for parameters  $\Delta_r$  to be determined later).
3.  $G_t^r$  is maintained explicitly by the algorithm with an adjacency-list access for every vertex.

We start by defining the three main collections of vertices of  $V(G)$ ,  $\mathcal{H}_t := \{H_t^1, \dots, H_t^R\}$ ,  $\mathcal{I}_t := \{I_t^1, \dots, I_t^R\}$ , and  $\mathcal{L}_t := \{L_t^1, \dots, L_t^R\}$  used in our algorithm (when clear from the context, or irrelevant, we may drop the subscript  $t$  from these sets). For simplicity of notation, we also define  $H_t^0 = I_t^0 = \emptyset$  and  $L_t^0 = V(G)$  for all  $t$ . We design these sets carefully in the next section to satisfy the properties below.

**PROPOSITION 5.1.** *At any time step  $t$ :*

1. The sets in  $\mathcal{H}_t \cup \mathcal{I}_t$ , i.e.,  $H_t^1, \dots, H_t^R, I_t^1, \dots, I_t^R$ , are all pairwise disjoint.
2. The sets in  $\mathcal{L}_t$  are nested, i.e.,  $L_t^1 \supseteq L_t^2 \supseteq \dots \supseteq L_t^R$ .
3. For any fixed  $r \in [R]$ ,  $H_t^r, I_t^r, L_t^r \subseteq L_t^{r-1}$  and partition  $L_t^{r-1}$ .

For any  $r \in [R]$ , the level- $r$  graph  $G_t^r$  is defined as the induced subgraph of  $G_t$  on  $L_t^r$ , i.e.,  $G_t^r := G_t[L_t^r]$ . Moreover,  $\mathcal{M}_t^r$  would be chosen carefully from the graph  $G_t^{r-1}$  such that  $\mathcal{M}_t^r \subseteq H_t^r$ .  $\mathcal{M}_t^*$  would also be an MIS of the graph  $G_t^R$ . We further have,

**PROPOSITION 5.2.** *At any time step  $t$ :*

1. For any  $r \in [R]$ , the independent set  $\mathcal{M}_t^r$  is an MIS of  $G_t[H_t^r]$ .
2. For any  $r \in [R]$ ,  $I_t^r$  is incident to some vertex of  $\mathcal{M}_t^r$  and  $L_t^r$  has no neighbor in  $\mathcal{M}_t^r$ .

Before we move on from this section, we show that Proposition 5.1 and 5.2 imply the Part-(1) of Invariant 4.

*Proof.* [Proof of Part-(1) in Invariant 4] Firstly, it follows from Part-(1) and Part-(3) of Proposition 5.1 that  $\mathcal{H}_t \cup \mathcal{I}_t \cup L_t^R$  partitions  $V(G) (= L_t^0)$ .

By Part-(1) of Proposition 5.2,  $\mathcal{M}_t^1$  is an MIS of  $G_t[H_t^1]$  and is also incident to all vertices in  $I_t^1$ . Hence, the only vertices not incident to  $\mathcal{M}_t^1$  are inside  $G_t^1$ .  $\mathcal{M}_t^2$  is not incident to any vertex of  $\mathcal{M}_t^1$  as  $\mathcal{M}_t^2 \subseteq H_t^2 \subseteq L_t^1$  and hence by Part-(2) of Proposition 5.2 are not incident to  $\mathcal{M}_t^1$ . We can hence continue as before and argue that any vertex not incident to  $\mathcal{M}_t^2$  can only belong to  $G_t^2$ . Repeating this argument for all  $r \in [R]$ , we obtain that  $\mathcal{M}_t^1 \cup \dots \cup \mathcal{M}_t^R$  are all an independent set in  $G_t$  and moreover, the only vertices not incident to them are in  $G_t^R$ . Since  $\mathcal{M}_t^*$  is an MIS of  $G_t^R$ , we obtain that  $\mathcal{M}_t^1 \cup \dots \cup \mathcal{M}_t^R \cup \mathcal{M}_t^*$  is an MIS of  $G_t$ . ■

### Level- $r$ Phases and Preprocessing Steps

We now construct the sets  $\mathcal{H}, \mathcal{I}, \mathcal{L}$ , plus the independent sets  $\mathcal{M}^1, \dots, \mathcal{M}^R$  from the previous section. These are defined through the notion of a level- $r$  phases for any  $r \in [R]$ . Each level- $r$  phase is responsible for maintaining the graph  $G_t^r$  and independent set  $\mathcal{M}_t^r$  defined in the previous section. A level- $r$  phase starts at some time step  $t_{\text{start}}^r$  and terminates at some time step  $t_{\text{end}}^r$  (we define the criteria for terminating a time step later) upon which the next level- $r$  phase starts. Both  $t_{\text{start}}^r, t_{\text{end}}^r \in [t_{\text{start}}^{r-1}, t_{\text{end}}^{r-1}]$ , i.e., any level- $r$  phase happens during a level- $(r-1)$  phase and it is possible that multiple level- $r$  phases start and terminate during a single level- $(r-1)$  phase. We now define the process during each phase.

Pick  $R$  probability parameters  $p_1, \dots, p_R \in (0, 1)$  such that  $p_r \geq 2 \cdot p_{r-1}$  for all  $r > 1$  and  $p_1 \geq \frac{1}{n}$ . We optimize the values of  $p_1, \dots, p_R$  at the end of the section to obtain the best bound possible from our nested approach. Moreover, we define  $\Delta_r := (5p_r^{-1} \cdot \ln n)$  for all  $r \in [R]$  (recall that  $\Delta_r$  is used in Part-(2) of Invariant 4).

At the beginning of a level- $r$  phase, we remove all vertices  $H^r, H^{r+1}, \dots, H^R$  (similarly for  $I$ - and  $L$ -vertices), as well as graphs  $G^r, G^{r+1}, \dots, G^R$ , and corresponding independent sets  $\mathcal{M}^r, \dots, \mathcal{M}^R$ , and  $\mathcal{M}^*$ . All these sets and graphs are then redefined through the following *preprocessing step*.

**LevelPreProcess( $r$ )** (preprocessing for level- $r$  phases):

1. Let  $t_0 := t_{\text{start}}^r$  denote the current time step.

All graphs and sets below are with respect to time  $t_0$  and hence we omit this subscript.

2. Let  $\tilde{H}^r$  be a set chosen by picking each vertex in  $V(G)$  independently w.p.  $p_r$ .
3. Define  $H^r := \tilde{H}^r \cap L^{r-1}$  and compute  $\mathcal{M}_{H^r} := \text{Greedy}(G[H^r])$ .
4. If  $r \leq R$ , perform `LevelPreProcess`( $r + 1$ ).

We note that at first glance it might not be clear that why we pick the set  $H^r$  from a larger domain in `LevelPreProcess`, but then only focus on  $H^r$  as the intersection of  $\tilde{H}^r$  with  $L^{r-1}$  (we could have just picked  $H^r$  by sampling each vertex in  $L^{r-1}$  w.p.  $p$ ). However, we also use the sets  $\tilde{H}^r$  crucially in our algorithm to detect whether the current phase should be terminated or not (for reasons which would become evident shortly) and hence we perform this rather counterintuitive sampling step. We now define the sets  $\mathcal{H}_t, \mathcal{I}_t, \mathcal{L}_t$  plus the independent sets  $\mathcal{M}_t^1, \dots, \mathcal{M}_t^R$  for all time steps  $t \in [t_{\text{start}}^r, t_{\text{end}}^r]$  as follows:

- $\mathcal{M}_t^r$  is defined to be  $\mathcal{M}_{H^r}$  defined in `LevelPreProcess` throughout the whole phase ( $\mathcal{M}_t^r$  is fixed during a level- $r$  phase).
- $H_t^r \in \mathcal{H}_t$  is defined to be  $H^r$  defined in `LevelPreProcess` throughout the whole phase ( $H_t^r$  is fixed during a level- $r$  phase).
- $I_t^r \in \mathcal{I}_t$  is defined to be any vertex in  $L_t^{r-1}$  which is not in  $H_t^r$  and is incident to  $\mathcal{M}_t^r$  in the graph  $G_t$  ( $I_t^r$  can vary during a level- $r$  phase).
- $L_t^r \in \mathcal{L}_t$  is defined to be any vertex in  $L_t^{r-1}$  which is neither in  $H_t^r$  nor in  $I_t^r$  in the graph  $G_t$  ( $L_t^r$  can vary during a level- $r$  phase).

We now define the termination criteria of a level- $r$  phase using the following time steps.

- $t_H^r \geq t_{\text{start}}^r$ : the first time step  $t$  where the updated edge  $e_t := (u_t, v_t)$  is such that  $u_t, v_t \in \tilde{H}^1 \cup \dots \cup \tilde{H}^r$ , and at least one of  $u_t$  or  $v_t$  belongs to  $\tilde{H}^r$ .
- $t_I^r \geq t_{\text{start}}^r$ : the first time step  $t$  in which the total number of times (since  $t_{\text{start}}^r$ ) that vertices in  $\tilde{H}^1 \cup \dots \cup \tilde{H}^r$  have been incident to an update reaches  $p_r^{-1}$ .
- $t_L^r \geq t_{\text{start}}^r$ : the first time step  $t$  in which  $\Delta(G_t[L_t^r]) > \Delta_r$ .

- $t_{\text{end}}^r := \min \{t_{\text{end}}^{r-1}, t_H^r, t_I^r, t_L^r, t_{\text{start}}^r + T_r\}$  where  $T_r := \frac{1}{24p_r^2}$ : the time step in which we terminate this phase (in other words, if any of the conditions above happens, the level- $(r - 1)$  that the current level- $r$  phase belongs to terminate, or we simply spend  $T_r$  updates in this phase, the phase finishes and the next one starts).

We first prove that by the criteria imposed for terminating each phase, the properties Propositions 5.1 and 5.2 are satisfied. We start with the simpler proof.

*Proof.* [Proof of Proposition 5.1]

For simplicity, we drop the subscript  $t$  from all sets below.

1.  $H^r, I^r, L^r$  are disjoint for each  $r \in [R]$  by definition. Moreover,  $H^r \cup I^r \subseteq L^{r-1}$ , while  $H^{r-1}, I^{r-1}$  are disjoint from  $L^{r-1}$  by definition. This means that  $H^r, I^r$  are also disjoint from any other set  $H^{r'}, I^{r'}$  for  $r \neq r'$ .
2. Each  $L^r$  is defined as a subset of vertices of  $L^{r-1}$ , hence  $L^r \subseteq L^{r-1}$ .
3. The disjointness of  $H^r, I^r, L^r$  is by definition. Also, by definition, we have  $I^r \cup L^r = L^{r-1} \setminus H^r$ , and hence the sets partition  $L^{r-1}$  at any time step. ■

*Proof.* [Proof of Proposition 5.2] For simplicity, we drop the subscript  $t$  from all sets below.

1. By definition of  $t_H^r$ , we always terminate a level- $r$  phase and start a new one if the update involved two vertices in  $\tilde{H}^1 \cup \dots \cup \tilde{H}^r$  with at least one of them in  $\tilde{H}^r$ . As  $H^r$  is a subset of  $\tilde{H}^r$ , this means that if an edge with both endpoints in  $H^r$  are updated, then we terminate this phase and start a new one. Otherwise, by definition, we have  $G_{t_{\text{start}}^r}[H^r] = G_t[H^r]$  for any  $t < t_H^r$ . Since  $\mathcal{M}_t^r = \mathcal{M}_{H^r}$  was an MIS of  $G_{t_{\text{start}}^r}[H^r]$ , this means that it is also an MIS of  $G_t[H^r]$ , proving this part.
2. This part follows from definition of  $\mathcal{M}_t^r = \mathcal{M}_{H^r}$  and the sets  $I^r$  and  $L^r$ . ■

We now use these properties to prove Part-(2) of Invariant 4.

*Proof.* [Proof of Part-(2) of Invariant 4] Recall that  $G_t^r := G_t[L_t^r]$ . By definition of the time step  $t_L^r$ , we always start a new level- $r$  phase whenever  $\Delta(G_t^r) > \Delta_r$ . As such, throughout the algorithm we always have that  $\Delta(G_t^r) \leq \Delta_r$ . ■

We also prove the following two auxiliary claims that are used in the rest of the proof.

CLAIM 5.1. *Let  $e_t := (u_t, v_t)$  be an update during a level- $r$  phase after which  $v_t$  needs to join or leave  $L_t^r$ . Then  $u_t \in H_t^1 \cup \dots \cup H_t^r$ .*

*Proof.* By Proposition 5.1,  $H_t^1, \dots, H_t^r, I_t^1, \dots, I_t^r, L_t^r$  partition  $V(G)$ . If both  $u_t, v_t \in L_t^r$ , this update cannot force  $v_t$  to leave  $L_t^r$ . Moreover, if  $u_t$  is in  $I_t^1 \cup \dots \cup I_t^r$ , then deleting or adding this edge does not change the set  $L_t^r$  (recall that  $L_t^r$  and  $I_t^r$  are defined with respect to  $H_t^r$  and are both subsets of  $L_t^{r-1}$ ). As such, the only way for  $v_t$  to join or leave  $L_t^r$  is if  $u_t$  belongs to  $H_t^1 \cup \dots \cup H_t^r$ , finalizing the proof (note that we assumed this update is happening during a phase and hence none of level-1 to level- $r$  phases are terminated which naturally change the definition of  $L_t^r$ ). ■

CLAIM 5.2. *Let  $t$  be any time step in  $[t_{\text{start}}^r, t_H^r)$ . Then  $\tilde{H}^r \cap L_t^{r-1} = H^r$ .*

*Proof.* Recall that  $H^r := \tilde{H}^r \cap L_{t_{\text{start}}^r}^{r-1}$  and since  $L_t^{r-1}$  can vary from  $L_t^{r-1}$  throughout the phase, a-priori it is not clear that  $H^r$  remains the same. However, for  $H^r$  to be different from  $\tilde{H}^r \cap L_t^{r-1}$ , a vertex in  $\tilde{H}^r$ , say  $v$ , should join or leave  $L_t^{r-1}$ . Consider the first time step  $t' \leq t$  such that  $v$  did this change and let  $(u_{t'}, v_{t'})$  be the updated edge at this time step. By Claim 5.1,  $u_{t'}$  should belong to  $H_{t'}^1 \cup \dots \cup H_{t'}^{r-1} \subseteq \tilde{H}_{t'}^1 \cup \dots \cup \tilde{H}_{t'}^{r-1}$ . But we also have that  $v_{t'} \in \tilde{H}_{t'}^r$ . This, by definition of  $t_H^r$  implies that  $t' = t_H^r$ , contradicting the choice of  $t < t_H^r$ . ■

We conclude this part by remarking that definition of the time step  $t_H^r$  immediately implies the following invariant.

INVARIANT 5. *The total number of updates during a level- $r$  phase that are incident to some vertex in  $\tilde{H}_t^1 \cup \dots \cup \tilde{H}_t^r$  is  $O(p_r^{-1})$ .*

**Successful Phases.** A level- $r$  phase is considered *successful* iff  $t_{\text{end}}^r = \min\{t_{\text{end}}^{r-1}, t_{\text{start}}^r + T_r\}$ . The following lemma is analogous to Lemma 3.1 in Section 3.

LEMMA 5.3. *For any  $r \in [R]$ , any given level- $r$  phase is successful with probability at least  $1/2$ .*

*Proof.* We calculate the probability that  $t_{\text{end}}^r < t_{\text{start}}^r + T_r$ . Recall that the adversary is non-adaptive and oblivious and hence we can fix the updates the adversary.

CLAIM 5.4.  $\Pr(t_H^r < t_{\text{start}}^r + T_r) \leq \frac{1}{6}$ .

*Proof.* For any  $t \geq t_{\text{start}}^r$ , let  $e_t := (u_t, v_t)$  denote the edge updated by the adversary at time  $t$ . Define  $\mathcal{E}_1(e_t)$  as the event that both  $u_t$  and  $v_t$  belong to  $\tilde{H}^1 \cup \dots \cup \tilde{H}^r$  and at least one of them belong to  $\tilde{H}^r$ . Consider the randomness in the choice of  $\tilde{H}^1 \cup \dots \cup \tilde{H}^r$ . We have,

$$\Pr(\mathcal{E}_1(e_t)) \leq 2p_r \cdot (p_1 + \dots + p_r) \leq 2p_r \cdot 2p_r = 4p_r^2,$$

where we used the fact that  $p_{r'} \geq 2p_{r'-1}$  for all  $r' \in [R]$  and hence  $p_1 + \dots + p_r \leq 2p_r$ . For any  $t \in [t_{\text{start}}^r, t_{\text{start}}^r + T)$ , define an indicator random variable  $X_t$  which is 1 iff  $\mathcal{E}_1(e_t)$  happens. Let  $X := \sum_t X_t$ . As such,

$$\begin{aligned} \Pr(t_H^r < t_{\text{start}}^r + T_r) &\leq \Pr(X \geq 1) \leq \mathbb{E}[X] \\ &= (T_r - 1) \cdot 4p_r^2 \leq \frac{1}{24p_r^2} \cdot 4p_r^2 \\ &= \frac{1}{6}, \end{aligned}$$

where the second inequality is by Markov bound. ■

CLAIM 5.5.  $\Pr(t_I^r < t_{\text{start}}^r + T_r) \leq \frac{1}{6}$ .

*Proof.* For any  $t \geq t_{\text{start}}^r$ , let  $e_t := (u_t, v_t)$  denote the edge updated by the adversary at time  $t$ . Define  $\mathcal{E}_2(e_t)$  as the event that at least one of the endpoints  $e_t$  belong to  $\tilde{H}^1 \cup \dots \cup \tilde{H}^r$ . Consider the randomness in the choice of  $\tilde{H}^1 \cup \dots \cup \tilde{H}^r$ . We have,

$$\Pr(\mathcal{E}_2(e_t)) \leq 2(p_1 + \dots + p_r) \leq 4p_r,$$

where we used the fact that  $p_1 + \dots + p_r \leq 2p_r$ . For any  $t \in [t_{\text{start}}^r, t_{\text{start}}^r + T)$ , define an indicator random variable  $Y_t$  which is 1 iff  $\mathcal{E}_2(e_t)$  happens. Let  $Y := \sum_t Y_t$ . We have,

$$\begin{aligned} \Pr(t_I < t_{\text{start}} + T) &\leq \Pr(Y \geq p_r^{-1}) \leq \mathbb{E}[Y] \cdot p_r \\ &\leq (T_r - 1) \cdot 4p_r \cdot p_r \\ &\leq \frac{1}{24p_r^2} \cdot 4p_r^2 = \frac{1}{6}, \end{aligned}$$

where the second inequality is by Markov bound. ■

CLAIM 5.6.  $\Pr(t_L^r < t_{\text{start}}^r + T_r) \leq \frac{1}{n^2}$ .

*Proof.* Let  $t_0 = t_{\text{start}}^r$  as in LevelPreProcess. First consider the graph  $G_{t_0}[L_{t_0}^{r-1}]$ . The set  $H^r$  chosen in LevelPreProcess is a set of vertices each chosen with probability  $p_r$  from  $L_{t_0}^{r-1}$ . Hence, by Lemma 2.2, by choice of  $S = H^r$  and  $U = L_{t_0}^r$ , and since  $\mathcal{M}_{H^r} = \text{Greedy}(G[H^r])$ , we have that  $\Delta(G_{t_0}[L_{t_0}^r]) \leq 5p_r^{-1} \ln n$ .

Now consider any time step  $t > t_0$ . Define the graph  $G'_t$  where  $G'_t[H^r] = G_{t_0}[H^r]$  and  $G'_t$  is equal to  $G_t$  everywhere else (i.e., we replaced the induced subgraph of  $G_t$  on vertices in  $H^r$  with the corresponding part in  $G_{t_0}$ ). By definition,  $\mathcal{M}_{H^r}$  is equal to  $\text{Greedy}(G'_t[H^r])$  ( $= \text{Greedy}(G_{t_0}[H^r])$ ). The problem with applying the above argument directly for  $t$  as well is that the set of vertices in  $L_t^{r-1}$  may have changed since  $L_{t_0}^{r-1}$  and when we chose  $H^r$  (i.e., the set of vertices of  $G'_t$  is not the same as  $G_{t_0}$ ). However, consider instead the set  $H' := \tilde{H}^r \cap L_t^{r-1}$ : these are again vertices chosen by picking each vertex of  $L_t^{r-1}$  with probability  $p_r$  (by definition of  $\tilde{H}^r$ ). By Claim 5.2,  $H' = H^r$  during one phase. As such,  $H^r = H'$ , and we can apply Lemma 2.2 as before and obtain that for the graph  $G'_t$ , with probability  $1 - 1/n^4$ ,  $\Delta(G'_t[L_t^r]) \leq 5p_r^{-1} \ln n$ . But since  $G'_t[L_t] = G_t[L_t]$  by definition, it follows that  $\Delta(G_t[L_t]) \leq 5p_r^{-1} \ln n$  as well. Taking a union bound over these  $\leq n^2$  graphs finalizes the proof. ■

By applying union bound to Claims 5.4, 5.5, and 5.6, the probability that  $\min\{t_H^r, t_I^r, t_L^r\} < t_{\text{start}}^r + T_r$  is at most  $1/6 + 1/6 + 1/n^2 \leq 1/2$ . This concludes the proof of Lemma 5.3. ■

**The Update Algorithm** We now show how to process the updates during different phases of the algorithm, and prove Part-(3) of Invariant 4.

**Processing Updates for a Level- $r$  Phase.**

Recall that each level- $r$  phase is mainly responsible for maintaining the graph  $G_t^r := G_t[L_t^r]$ . We show how to do this in the following. Let  $e_t := (u_t, v_t)$  be the updated edge. Recall that by Proposition 5.1,  $H_t^1 \cup \dots \cup H_t^r \cup I_t^1 \cup \dots \cup I_t^r \cup L_t^r$  partitions the set of vertices  $V(G)$ , and  $H_t^{r'} \subseteq \tilde{H}_t^{r'}$  for all  $r' \in [R]$ . Finally, we note that we process the updates according to the ordering below and when some updates can be possibly processed according to two or more of the cases below, we always update it according to the first case it appears.

- **Case 1.** Updates that immediately terminate this phase:
  - **Case 1-a.** Both  $u_t$  and  $v_t$  belong to  $\tilde{H}_t^1 \cup \dots \cup \tilde{H}_t^r$ . These updates by definition of  $t_H^r$  either terminate the level- $r$  phase directly, or terminate some level- $r'$  phase for  $r' \leq r$ , and hence indirectly terminate the current level- $r$  phase. These updates are then processed after restarting the level- $r'$  phase (and all phases inside it).
  - **Case 1-b.** Any update that result in time steps  $t_I^r, t_L^r, t_{\text{end}}^{r-1}$ . These updates are again

processed after restarting the current phase and in the next phase. Note that deciding whether an update can result in either of these events can be easily detected in  $O(1)$  time per each update.

- **Case 2.** Updates that do not change the set  $L_t^r$  (hence do not change vertices of  $G_t^r$  but can potentially update its edges):
  - **Case 2-a.** Both  $u_t$  and  $v_t$  belong to  $I_t^1 \cup \dots \cup I_t^r$ . There is nothing to do in this case.
  - **Case 2-b.** Both  $u_t$  and  $v_t$  belong to  $L_t^r$ . We only need to update the edge in  $(u_t, v_t)$  in the graph  $G_t^r := G_t[L_t^r]$  which can be done in  $O(1)$  time trivially.
  - **Case 2-c.**  $u_t$  belongs to  $I_t^1 \cup \dots \cup I_t^r$  and  $v_t$  belongs to  $L_t^r$  (or vice versa). There is nothing to do in this case either.
- **Case 3.** Updates that can (potentially) change the set  $L_t^r$  (and hence the vertices of  $G_t^r$ ); recall that by Claim 5.1, one endpoint of any such update needs to be in  $H_t^1 \cup \dots \cup H_t^r$ :
  - **Case 3-a.**  $u_t$  is in  $H_t^1 \cup \dots \cup H_t^r$  and  $v_t$  is in  $I_t^1 \cup \dots \cup I_t^r$  (or vice versa). If  $e_t$  is inserted, no set needs to be changed. However, if  $e_t$  is deleted, it might be that  $v_t$  needs to be removed from  $I_t^{r'}$  and be inserted in either  $I_t^{r''}$  for some  $r' \leq r'' \leq r$  or to  $L_t^r$ . If it is to be inserted in  $I_t^{r''}$ , it necessarily means that it also needs to be inserted to the set  $L_t^{r''-1}$  and hence we process this update at a level- $r''$  phase as well and that phase then informs the next level phase in case it needs to also add  $v_t$  to its corresponding  $L$ -set and so on. As such, without loss of generality, in the current phase, we can focus on the case when  $v_t$  needs to be inserted to  $L_t^r$ .  
To do this, we iterate over all neighbors of  $v_t$  in the graph  $G_t^{r-1}$  and find all the ones that also belong to the set  $L_t^r$  (recall that  $L_t^r \subseteq L_t^{r-1}$  and  $L_t^{r-1}$  is the vertex-set of  $G_t^{r-1}$ ). This takes  $O(\Delta_{r-1})$  time as maximum degree of  $G_t^{r-1}$  is at most  $\Delta_{r-1}$  by Part-(2) of Invariant 4. Here, we also assumed inductively that Part-(3) of Invariant 4, holds for graphs  $G_t^1, \dots, G_t^{r-1}$ . We then insert this vertex to  $G_t^r$  and update the adjacency-list of all its neighbors in the graph  $G_t^r$  in  $O(\Delta_r)$  time. Finally, we pass this update to the next level phase to process (updates of this form are passed from higher level phases to lower level phases).

- **Case 3-b.**  $u_t$  is in  $H_t^1 \cup \dots \cup H_t^r$  and  $v_t$  is in  $L_t^r$  (or vice versa). If  $e_t$  is deleted, no sets need to be changed. However, if  $e_t$  is inserted, it might be that  $v_t$  needs to leave  $L_t^r$  and join  $I_t^{r'}$  for some  $r' \leq r$ . We first delete  $v_t$  with all its incident edges from  $G_t^r$  using the adjacency-list representation we maintained for this graph. This takes  $O(\Delta_r)$  time as the maximum degree of  $G_t^r$  is at most  $\Delta_r$  by Part-(2) of Invariant 4. If this vertex needs to be inserted to  $I_t^r$  we do so, otherwise there is nothing to do (note that, this update is being processed by *all* level- $r'$  phases for  $r' \leq r$  and the corresponding level that needs to insert  $v_t$  to  $I_t^{r'}$  would do so).

One can verify that cases above contain all possible updates. This immediately proves Part-(3) of Invariant 4.

**Processing Updates to Maintain  $\mathcal{M}_t^*$ .** Recall that we also need to maintain  $\mathcal{M}_t^*$  which is an MIS of the graph  $G_t^R$ . To do this, we simply run the deterministic algorithm of Lemma 2.1 on the graph  $G_t^R$  which we are explicitly maintaining by Part-(3) of Invariant 4. As this deterministic algorithm can handle vertex-insertions and deletions as well as edge insertions and deletions,  $\mathcal{M}_t^*$  would indeed be an MIS of  $G_t^R$  and this requires  $O(\Delta_R)$  amortized update time as maximum degree of  $G_t^R$  is  $\Delta_R$  by Part-(2) of Invariant 4.

We now bound the total running time of the algorithm responsible for each phase, as well as the one needed for maintaining  $\mathcal{M}_t^*$ .

**LEMMA 5.7.** *Let  $K$  denote the number of updates in a particular level- $r$  phase. The update algorithm for the level- $r$  phase maintains the independent set  $\mathcal{M}_t^r$  and graph  $G_t^r$  (deterministically) in  $O(n \cdot \Delta_{r-1} + K)$  time.*

*Proof.* The cost of bookkeeping the data structures in the update algorithm is  $O(1)$  per each update. The two main time consuming steps are hence the preprocessing done at the beginning of the level- $r$  phase and the cost of maintaining the graph  $G_t^r$  throughout the phase.

The preprocessing algorithm takes linear time in the graph it processes. As it is performed over  $G_{t_{\text{start}}}^{r-1}$  and maximum degree of  $G_t^{r-1} \leq \Delta_{r-1}$  throughout (by Invariant 4), the preprocessing step of a level- $r$  phase takes  $O(n \cdot \Delta_{r-1})$  time.

For the latter task, performing all updates except for Case 3 can be done in  $O(1)$  time per each update, while Case 3 updates require  $O(\Delta_{r-1})$  time per update as argued above. However note that by

Claim 5.1, any Case 3 update necessarily contains a vertex in  $H_t^1 \cup \dots \cup H_t^r \subseteq \tilde{H}_t^1 \cup \dots \cup \tilde{H}_t^r$ . By Invariant 5, the total number of such updates during a level- $r$  phase is at most  $p_r^{-1}$ . As such, the total time needed to process Case 3 phases is  $O(p_r^{-1} \cdot \Delta_{r-1})$  which is at most  $O(n \cdot \Delta_{r-1})$  as  $p_r^{-1} \leq p_1^{-1} \leq n$ . ■

**LEMMA 5.8.** *Let  $K$  denote the number of updates in a particular level- $R$  phase. The update algorithm maintains an MIS  $\mathcal{M}_t^*$  in  $G_t^R$  (deterministically) in  $O(n \cdot \Delta_R + K \cdot \Delta_R)$  time.*

*Proof.* Follows from Lemma 2.1 as by Invariant 4, maximum degree of  $G_t^R$  is at most  $\Delta_R$  and we only “start” the deterministic algorithm in Lemma 2.1, once per each level- $R$  phase. ■

**Proof of Theorem 5.1** We are now ready to prove Theorem 5.1. The correctness of the algorithm immediately follows from Lemmas 5.7 and 5.8 and Part-(1) of Invariant 4, hence, it only remains to bound the amortized update time of the algorithm.

Fix a sequence of  $K$  updates and for any  $r \in [R]$ , let  $P_1^r, \dots, P_{k_r}^r$  denote the different phases of the algorithm over this sequence (i.e., each  $P_i^r$  corresponds to the updates inside one level- $r$  phase). We compute the time spent by the overall algorithm in level- $r$  phase, as well as the algorithm for maintaining  $\mathcal{M}_t^*$  separately.

**Total Time Spent Across All Level- $r$  Phases.** By Lemma 5.7, the total time spent across all level- $r$  phases is  $O(k_r \cdot n \cdot \Delta_{r-1} + K)$  as  $K = \sum_i |P_i^r|$ . Hence, we only need to upper bound  $k_r$ .

**LEMMA 5.9.** *For any  $r \in [R]$ ,  $\mathbb{E}[k_r] = O(K \cdot p_r^2)$ .*

*Proof.* We prove the lemma by induction on  $r$ . For the base case, recall that a level-1 phase  $P_i^1$  is successful iff  $|P_i^1| = T_1 (= \frac{1}{24p_1^2})$ . The probability that  $P_i^1$  is successful is at least  $1/2$  by Lemma 5.3. Any successful phase includes  $T_1$  updates and hence we can have at most  $K/T_1$  successful level-1 phases (even if we assume the other phases include no updates). By the same argument as in Lemma 3.7, we have that  $\mathbb{E}[k_1] \leq 2K/T_1 = O(K \cdot p_1^2)$ .

We now prove the induction step. Recall that a level- $r$  phase  $P_i^r$  is successful iff the level- $(r-1)$  phase that contains it terminate, or  $|P_i^r| = T_r (= \frac{1}{24p_r^2})$ . The probability of being successful is also at least  $1/2$  by Lemma 5.3. Finally, note that at most  $k_{r-1}$  level- $r$  phases can terminate because the corresponding level- $(r-1)$  phase that contain them terminated (by definition of  $k_{r-1}$ ). The number of remaining successful phases are at most  $K/T_r$ . As such, by the above argument  $\mathbb{E}[k_r] \leq 2K/T_r + \mathbb{E}[k_{r-1}] = O(K \cdot p_r^2)$  by induction hypothesis as  $p_r \geq 2 \cdot p_{r-1}$ . ■



As such, the expected running time of this part is:

$$O(K \cdot n \cdot p_r^2 \cdot \Delta_{r-1} + K) = O(K \cdot \log^2 n) \cdot \left( n \cdot \frac{\Delta_{r-1}}{\Delta_r^2} \right),$$

by the choice of  $\Delta_r$  and  $p_r$  (note that  $n \cdot \Delta_{r-1} > \Delta_r^2$  for all  $r \in [R]$ ).

**Total Time Spent for Maintaining  $\mathcal{M}_t^*$ .** By Lemma 5.8, the total time spent for maintaining  $\mathcal{M}_t^*$  is  $O(k_R \cdot n \cdot \Delta_R + K \cdot \Delta_R)$ . As, by Lemma 5.9,  $\mathbb{E}[k_R] = O(K \cdot p_R^2)$ , we have that the expected running part of this time is:

$$O(K \cdot p_R^2 \cdot n \cdot \Delta_R + K \cdot \Delta_R) = O(K \cdot \log^2 n) \cdot \left( \frac{n}{\Delta_R} + \Delta_R \right).$$

**Total Running Time.** The total expected running time of the algorithm is now:

$$O(K \cdot \log^2 n) \cdot \left( \sum_{r=1}^R \left( \frac{n \cdot \Delta_{r-1}}{\Delta_r^2} \right) + \frac{n}{\Delta_R} + \Delta_R \right).$$

Recall that  $\Delta_r := 5p_r^{-1} \cdot \ln n$ . We pick the values of  $\Delta_1, \dots, \Delta_R$  (by choosing  $p_1, \dots, p_R$  in the algorithm) to optimize the above bound. By our assumption that  $p_R > 2p_{R-1}$ , we have that  $\frac{n \cdot \Delta_{R-1}}{\Delta_R^2} > \frac{n}{\Delta_R}$ . As such we can simplify the bound above to:

$$O(K \cdot \log^2 n) \cdot \left( \sum_{r=1}^R \left( \frac{n \cdot \Delta_{r-1}}{\Delta_r^2} \right) + \Delta_R \right).$$

To optimize this bound, we form the following equations:

$$(5.2) \quad \frac{n^2}{\Delta_1^2} = \frac{n\Delta_1}{\Delta_2^2} = \frac{n\Delta_2}{\Delta_3^2} = \dots = \frac{n\Delta_{R-1}}{\Delta_R^2} = \Delta_R.$$

One can then use all the equalities except for the last one to prove by induction that:

$$\Delta_i = \Delta_{i+1}^{\left( \frac{2^{i+1}-2}{2^{i+1}-1} \right)} \cdot n^{\left( \frac{1}{2^{i+1}-1} \right)}.$$

Then using the final equality in Eq (5.2), we obtain that:

$$\Delta_R = n^{\left( \frac{1}{2} \cdot \frac{2^R}{2^R-1} \right)} = O(\sqrt{n}),$$

where the second inequality is by the choice of  $R = 2 \log \log n$ , and thus having  $n^{\left( \frac{1}{2^R-1} \right)} = O(1)^3$ .

<sup>3</sup>We remark our algorithm in Section 3 can be seen as a special case of the algorithm in this section with parameter  $R = 2$  instead of  $R = 2 \log \log n$ . Using the calculation above, it is easy to see that for the choice of  $R = 2$ , the bound on  $\Delta_R$  is  $O(n^{2/3})$  as in the algorithm in Section 3 (note that we are measuring the log  $n$ -parameters outside this calculation).

All in all, this implies that the total expected running time of the algorithm is:

$$O(K \cdot \log^2 n) \cdot (R \cdot \sqrt{n}) = O(K \cdot \sqrt{n} \cdot \log^2 n \cdot \log \log n),$$

finalizing the proof of expectation-bound in Theorem 5.1.

To obtain the bound, with high probability, we can apply the same exact argument in Lemma 3.8 in Section 3 to Lemma 5.9; as the smallest value of  $p_r$  for  $r \in [R]$  belongs to  $p_1$  and it is equal to  $\Theta(n^{-3/4} \cdot \log n)$  (simply plug in the value of  $\Delta_R$  in the first term of Eq (5.2)), we obtain that as long as  $K = \Omega(n^{3/2} \log n)$ , we obtain the bound with high probability. For smaller values of  $K$ , we again do as in Section 3, and obtain that the total running time of the algorithm in this case is  $O(n^2 \log^3 n)$ , concluding the proof of Theorem 5.1.

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