Sublinear Algorithms for $(\Delta + 1)$ Vertex Coloring

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Joint work with Yu Chen (Penn) and Sanjeev Khanna (Penn)

Graph Coloring

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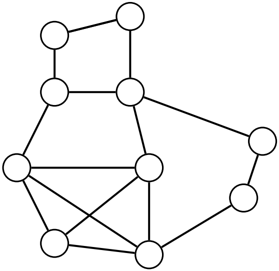

a graph G

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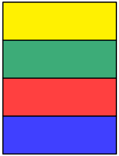

a palette of 4 colors

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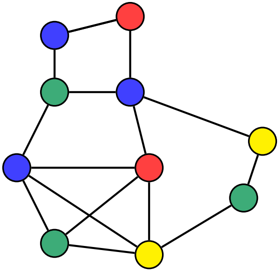

a proper 4-coloring of G

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Any partial coloring can be extended to a proper $(\Delta + 1)$ coloring.

Closely related to a plethora of other problems: maximal independent set, maximal matching, $(2\Delta - 1)$ edge coloring, $\cdots$

The Greedy Algorithm for $(\Delta + 1)$ Coloring

On a graph $G(V, E)$:

1. Iterate over vertices of V in arbitrary order,
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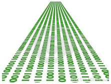

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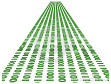
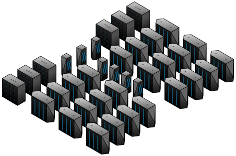


3. Massively parallel computation (MPC) algorithms:
   - Process the graph in a distributed fashion with limited communication.



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- "Exact" problems are typically hard for sublinear algorithms: one needs "approximation".

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Our algorithms are randomized:

- Output a $(\Delta + 1)$ coloring with high probability,
- Otherwise output FAIL.

Our Results: Sublinear Time Algorithms

The standard query model for dense graphs:

- Degree queries: what is degree of the vertex v?
- Pair queries: is (u, v) an edge?
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Prior Results:

No sublinear time algorithm for $(\Delta + 1)$ coloring.

Fastest algorithm: the greedy algorithm.

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- Queries are chosen non-adaptively.
- $\Omega(n\sqrt{n})$ query lower bound even for adaptive algorithms.

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- Process the stream in one pass and $\widetilde{O}(n)$ space.

Prior Results:

- No streaming algorithm for $(\Delta + 1)$ coloring with $o(n\Delta)$ space.
- Parallel to our work. Easier problem of $(\Delta + o(\Delta))$: a semi-streaming algorithm by [Bera and Ghosh, 2018].

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Semi-streaming algorithms:

- Edges are appearing one by one in a stream.
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- Our algorithm works even in dynamic graph streams.

Our Results: MPC Algorithms

MPC algorithms with near-linear memory per-machine:

- Edges are partitioned arbitrarily across multiple machines.
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Prior Results:

- An $O(\log\log\Delta \cdot \log^*(n))$ round algorithm with $\widetilde{O}(n)$ memory for $(\Delta+1)$ coloring [Parter, 2018].
- Parallel to our work. the round-complexity improved to $O(\log^*(n))$ rounds [Parter and Su, 2018].
- Easier problem of $(\Delta + o(\Delta))$ coloring: an $O(1)$ round algorithm with $n^{1+\Omega(1)}$ memory [Harvey et al., 2018].

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- Our algorithm only requires one round assuming public randomness.
- The first constant round MPC algorithm with $\widetilde{O}(n)$ memory for one of "classic four local distributed graph problems".

Our Main Result

The central tool: a structural result for $(\Delta + 1)$ coloring.

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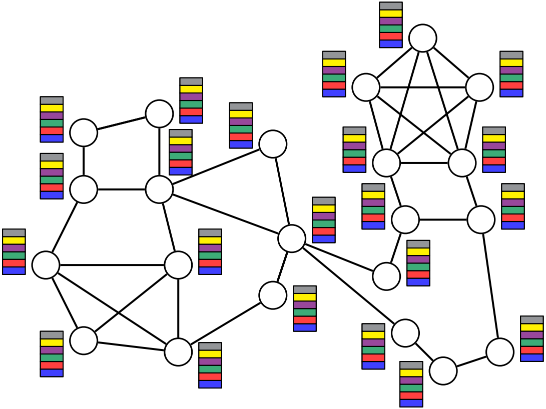
Palette Sparsification Theorem.

For every vertex v, sample $O(\log n)$ colors $L(v)$ from $\{1, \ldots, \Delta + 1\}$.

W.h.p., G can be colored by coloring any vertex v from the list $L(v)$.

Palette Sparsification: An Illustration



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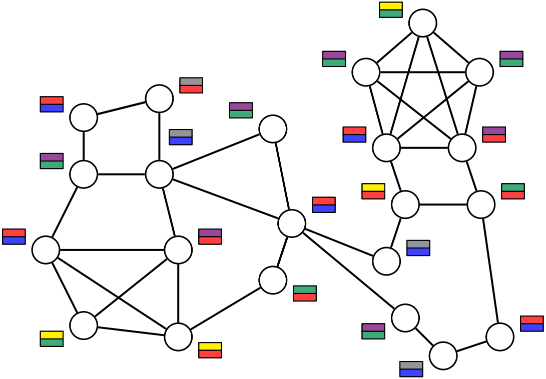

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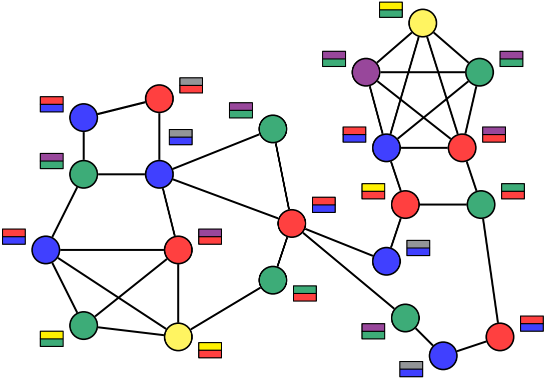

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$$n\Delta \cdot O(\log n) \cdot O(\frac{\log n}{\Delta}) = O(n \cdot \log^2 n).$$

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Non-adaptively sparsify a graph with $O(n\Delta)$ edges down to $\widetilde{O}(n)$ edges; still recover a proper $(\Delta + 1)$ coloring!

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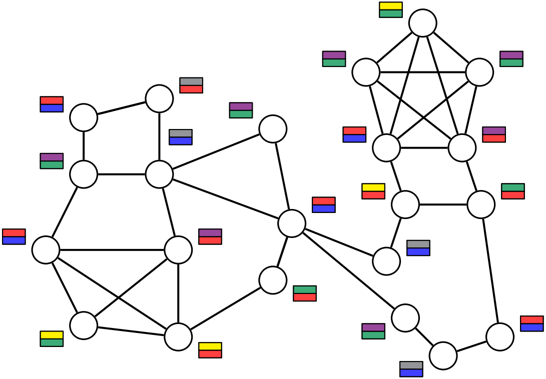

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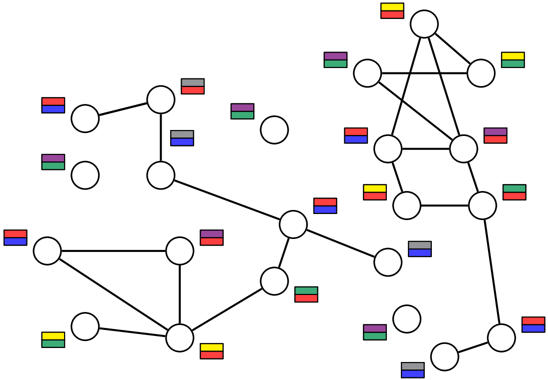

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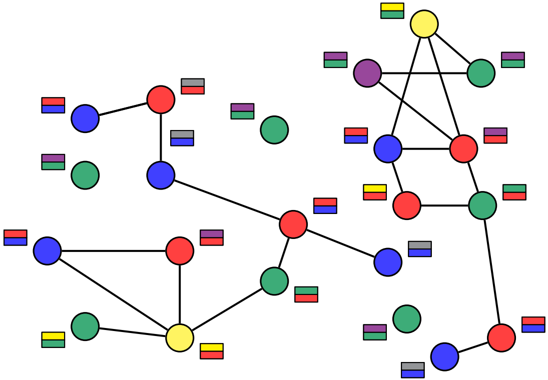

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A Slight Reformulation

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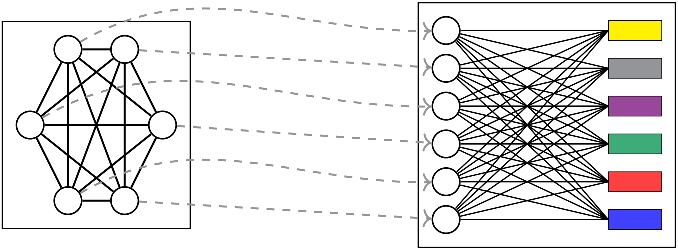

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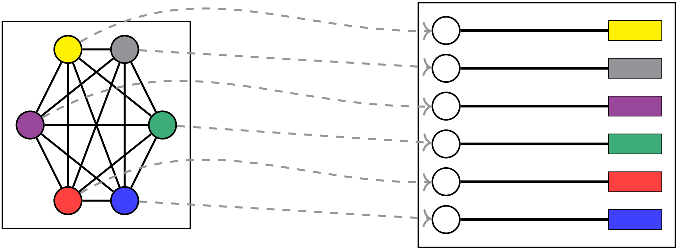

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$(\Delta + 1)$ **Coloring:** Finding a perfect matching in the palette graph.

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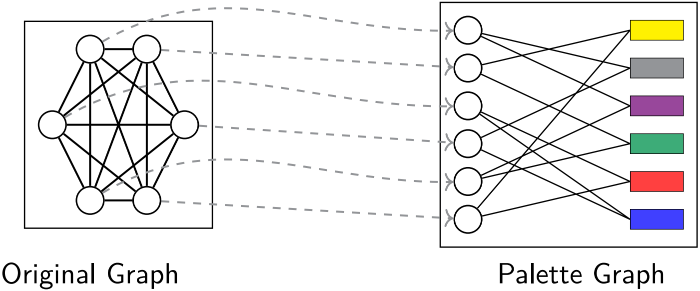

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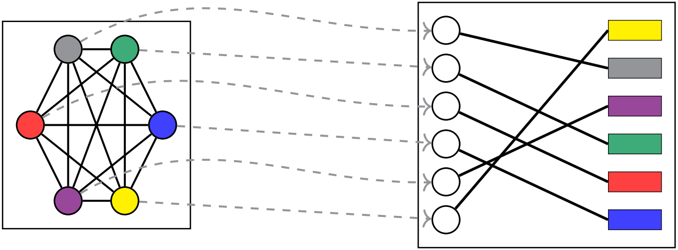

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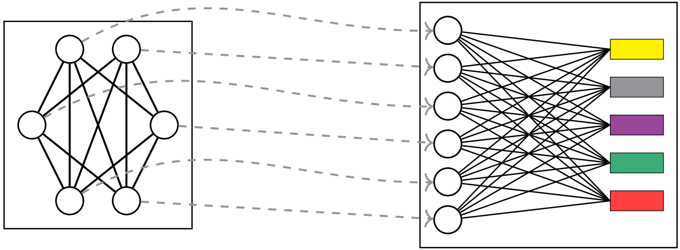

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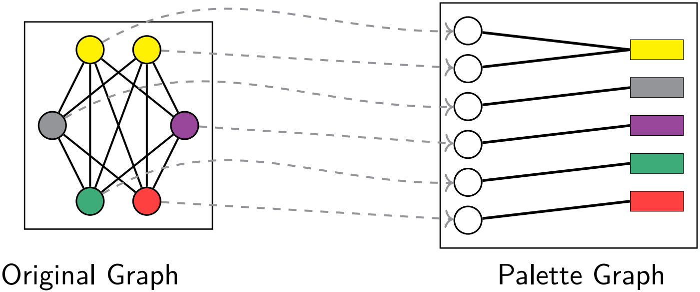

$(\Delta + 1)$ **Coloring:** Finding a "good" subgraph in the palette graph.

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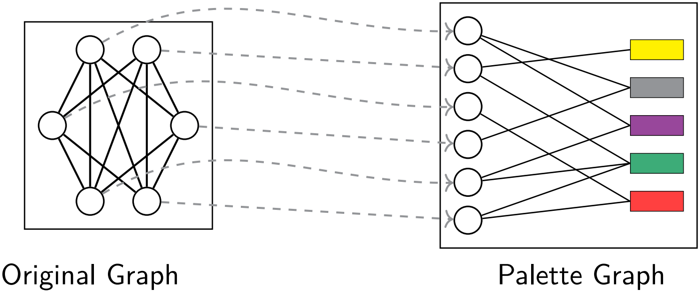

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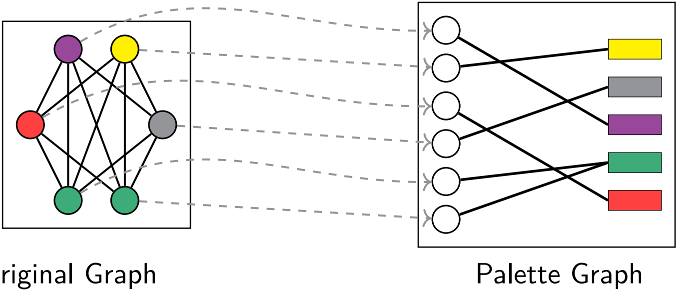

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But not that helpful for graphs that are "far from" cliques.

Handling Graphs that are Far From Cliques

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This proves the palette sparsification theorem for low degree graphs.

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Cliques

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Cliques ⟷ Low Degree Graphs

Assignment in random graphs — Direct simulation of greedy

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Our approach: Decompose the graph into dense and sparse regions, then apply the previous ideas to each part.

A Network Decomposition

We exploit and modify the decomposition of Harris, Schneider, and Su [Harris et al., 2016] for distributed $(\Delta + 1)$ coloring.

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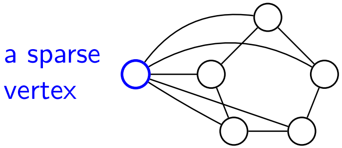

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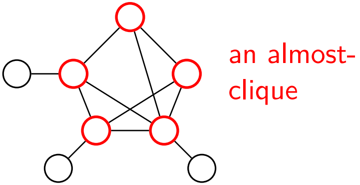

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   - Hard part: We need a generalization of ideas before in the assignment reformulation for almost-cliques.

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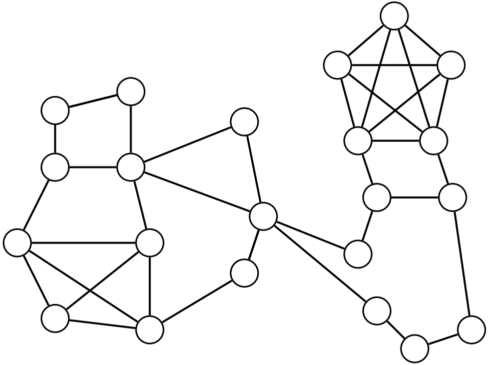

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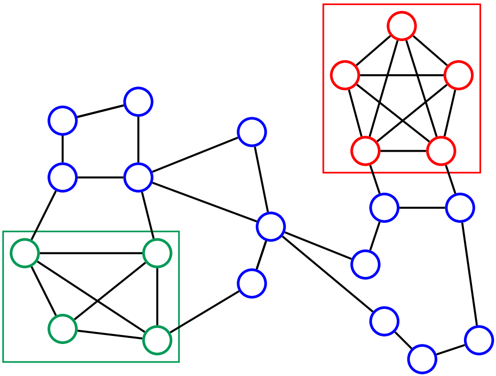

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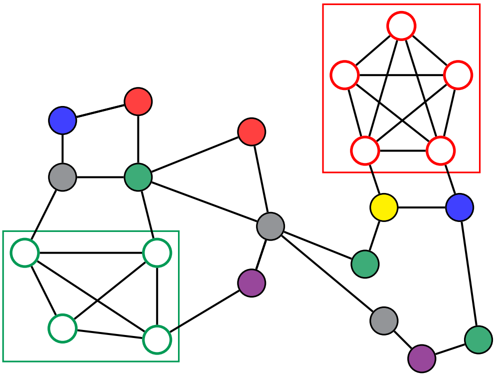

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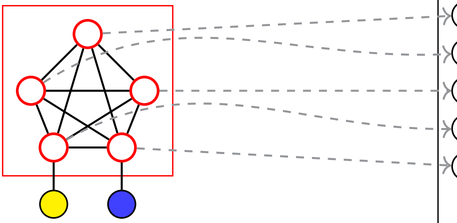
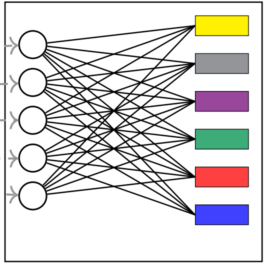


Almost-Clique



Palette Graph

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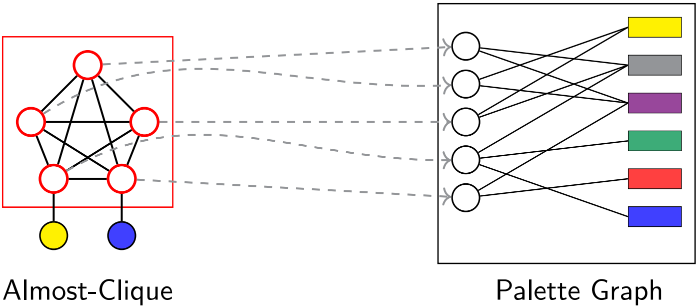

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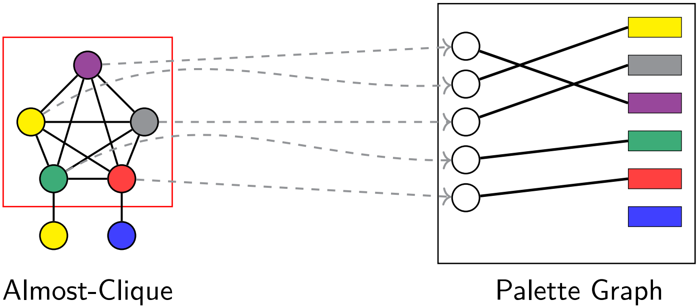

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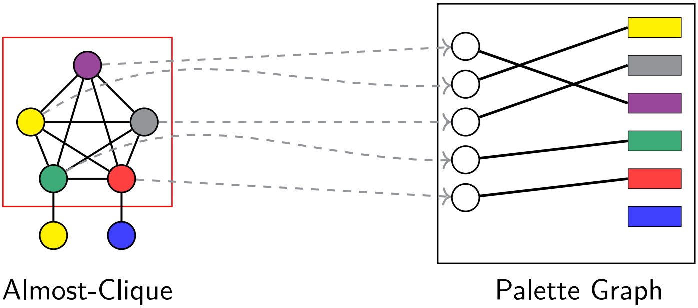

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Main challenge: vertices in an almost-clique may have some colored neighbors outside while the almost-clique may have size $> \Delta + 1$.

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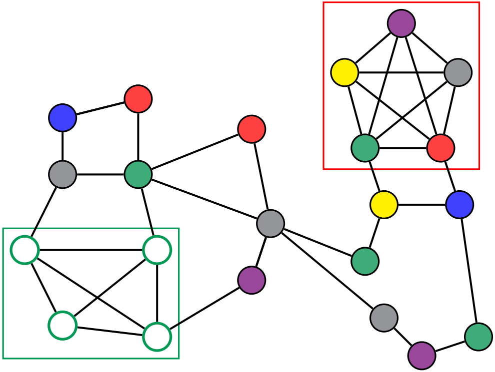

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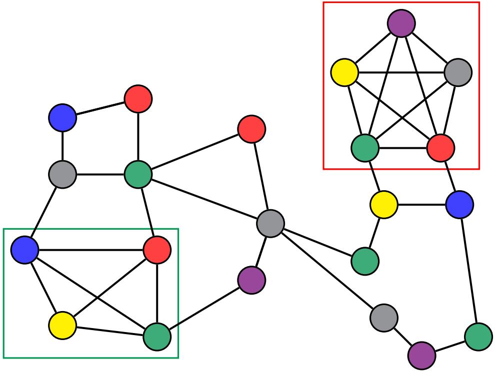

Sublinear Algorithms from Palette Sparsification Theorem

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This gives us our sublinear algorithms modulo a caveat...

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- We design sublinear algorithms for finding an approximate decomposition in each model.

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- Beyond greedy algorithms for sublinear algorithms: Can non-adaptive sparsification help other problems?

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