Randomized Composable Coreset for Matching and Vertex Cover

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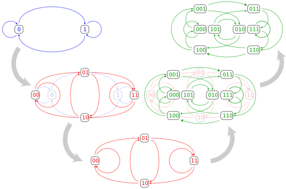
Joint work with Sanjeev Khanna (Penn)

Massive Graphs

Massive graphs abound in variety of applications: web graph, social networks, biological networks, etc.



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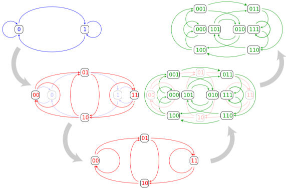

How to deal with computation over such massive graph inputs?

Distributed Computing

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Main measures of efficiency: communication cost and round complexity.

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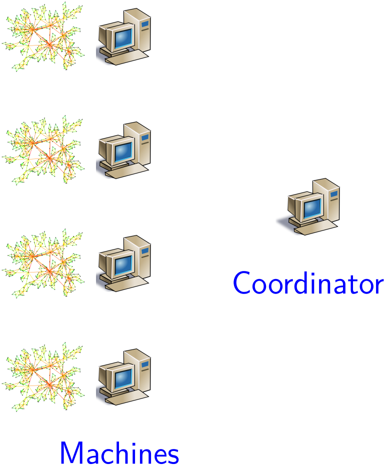

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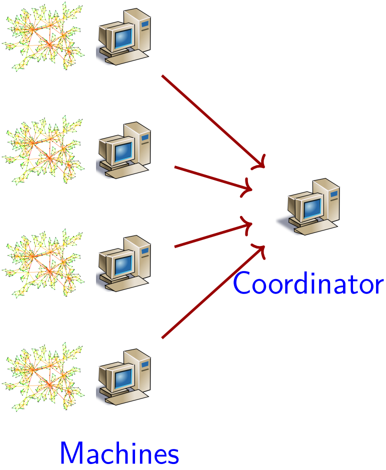

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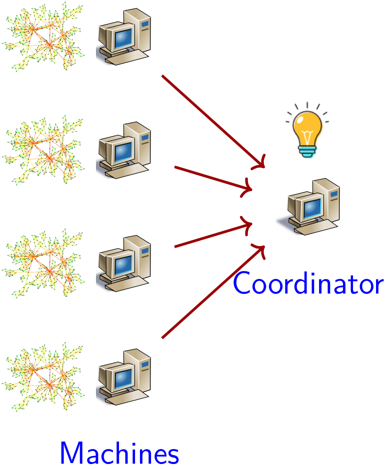
4. The coordinator computes the answer based on the summaries.



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3. Applications to other models of computation:
   - For example, lower bounds in dynamic streams.

Simultaneous Protocols

Many general techniques for designing simultaneous protocols, including:

- Linear sketches
- Composable coresets
- Mergable summaries
- Sampling

...

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Introduced for graph problems by Ahn, Guha, and McGregor [Ahn et al., 2012a].

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Many graph problems admit natural composable coresets; for instance, connectivity, sparsifiers, and spanners.

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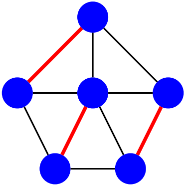
Maximum Matching and Minimum Vertex Cover

Matchings and Vertex Covers

- Matching: A collection of vertex-disjoint edges.



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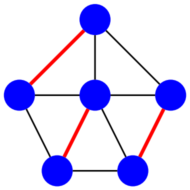
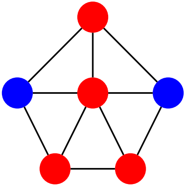


- Maximum Matching problem: Find a matching with a largest number of edges.

Matchings and Vertex Covers

- Vertex Cover: A collection of vertices containing at least one end point of every edge.



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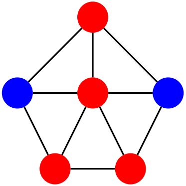

- Minimum Vertex Cover problem: Find a vertex cover with a smallest number of vertices.

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Can we distribute the original input in a better way?

Our Results in a Nutshell

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Our work is the first illustration in the domain of graph problems.

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Defined originally by [Mirrokni and Zadimoghaddam, 2015] in the context of distributed submodular maximization.

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This motivates a slightly more general notion of composable coresets.

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Size of a coreset: number of edges + number of specified vertices.

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Theorem

There exists an $O(\log n)$*-approximation randomized composable coreset of size* $O(n \cdot \log n)$ *for the vertex cover problem.*

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Remark. These bounds are tight for all values of α.

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Theorem

There exists simultaneous protocol with approximation guarantee

1. $O(1)$ *for the matching problem, and,*
2. $O(\log n)$ *for the vertex cover problem,*

that require only $\widetilde{O}(k \cdot n)$ *total communication when the input is randomly partitioned between* k *machines.*

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The number of rounds of a MapReduce algorithm usually determines the dominant cost of the computation.

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Any α-approximation simultaneous protocol (not necessarily a coreset) requires

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For adversarial partitions, an $\Omega(nk/\alpha^2)$ lower bound for matching was known previously even for protocols that are allowed multiple rounds of communication [Huang et al., 2015].

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We prove that $|M| = \Omega(\text{opt})$, where opt is the size of a maximum matching in G.

This implies that there exists an $O(1)$-approximate matching in $M_1 \cup \ldots \cup M_k$.

Analysis Sketch: A Key Lemma

Lemma

At any step $i \in [k]$, either the greedy matching is already of size $\Omega(\text{opt})$, or w.h.p., we can increase the size of the current matching by adding $\Omega(\text{opt}/k)$ edges from M_i greedily.

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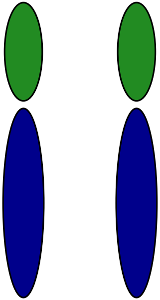
This immediately implies that the matching output by the greedy algorithm has size $\Omega(\mathsf{opt})$ w.h.p.

Proof Sketch

- Consider the set of $o(\text{opt})$ already matched vertices by the greedy algorithm.



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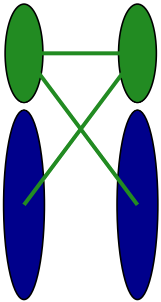

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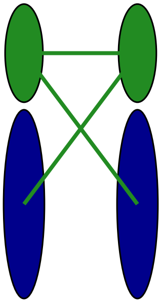
- Define μ_{old} as size of a maximum matching in $G^{(i)}$ using only edges in E_{old}.



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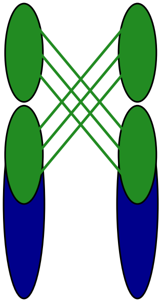

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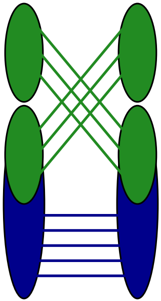

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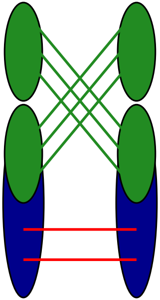
- Fix a maximum matching in E_{old}: at most $o(\text{opt})$ vertices that were previously unmatched are in the matching.
- Hence, G contains a matching of size $\Omega(\text{opt})$ outside the set of vertices matched by μ_{old}.
- By random partitioning, w.h.p., $\Omega(\text{opt}/k)$ such edges appear in $G^{(i)}$.



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- Fix a maximum matching in E_{old}: at most $o(\text{opt})$ vertices that were previously unmatched are in the matching.
- Hence, G contains a matching of size $\Omega(\text{opt})$ outside the set of vertices matched by μ_{old}.
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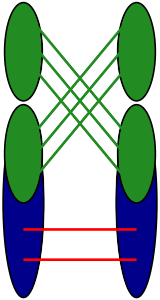

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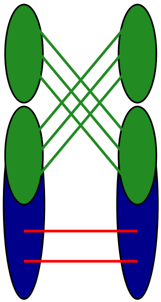

Corollary. Any maximum matching of $G^{(i)}$ contains $\Omega(\text{opt}/k)$ edges that can be added to the greedy matching.

Randomized Composable Coreset for Matching

We showed that,

Theorem

Any maximum matching is an $O(1)$-randomized composable coreset of size at most $n/2$ for the matching problem.

A Randomized Composable Coreset for Vertex Cover

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This peeling process was introduced originally by [Parnas and Ron, 2007] in the context of sublinear time algorithms.

A Randomized Coreset for Vertex Cover

The algorithm to compute the coreset on each machine $i \in [k]$:

1. Pick all vertices in $G^{(i)}$ with degree more than $n/2k$ and add them to the final vertex cover.

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Size of the coreset is clearly $O(n \cdot \log n)$.

Analysis Sketch

Define opt as size of a minimum vertex cover in G.

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We show that the set of all specified vertices across all coresets is of size $O(\mathsf{opt} \cdot \log n)$.

This finalizes the proof as any edge not covered by any of specified vertices is communicated in some coreset.

Analysis Sketch: A Key Lemma

Lemma

W.h.p. at most $O(\mathsf{opt} \cdot \log n)$ *vertices are specified to be added to the final vertex cover in total.*

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1. The peeling process is quite sensitive to the exact degrees.
2. Slight changes in the degree can move vertices across iterations, potentially leading to a cascading effect.

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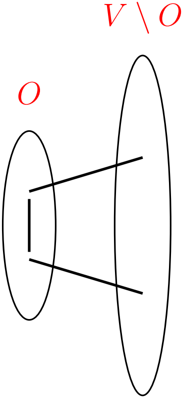
3. Show that the actual peeling process on each machine "faithfully" mimics this hypothetical process.

Proof Sketch

Define O as a minimum vertex cover of G. The hypothetical peeling process is as follows:



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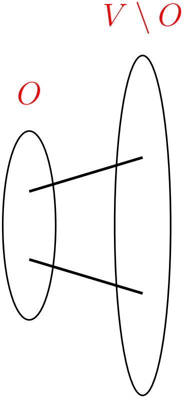

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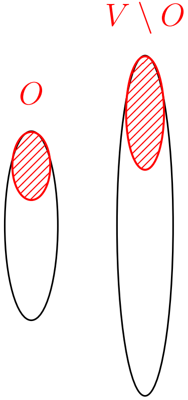

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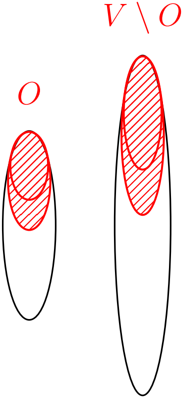

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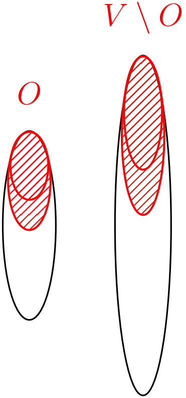
- Repeat the above process until the degree threshold reaches $\Theta(k \log n)$.



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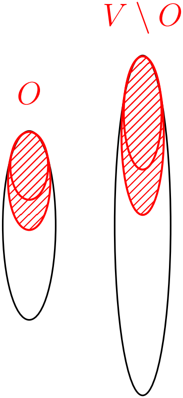
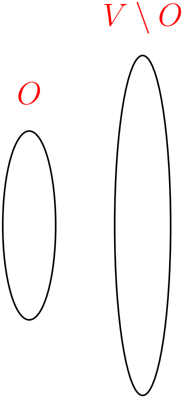


Claim. The number of peeled vertices from $V \setminus O$ in each iteration is at most $2\,|O|$.

Proof Sketch

Main Claim. For any machine $i \in [k]$ and any iteration of the peeling process:



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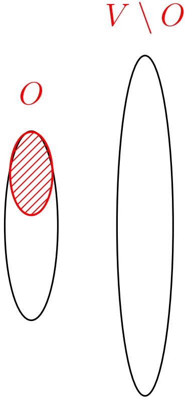

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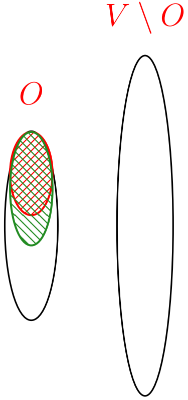

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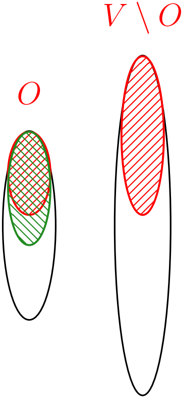

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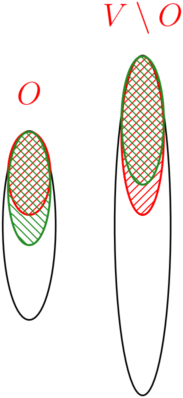

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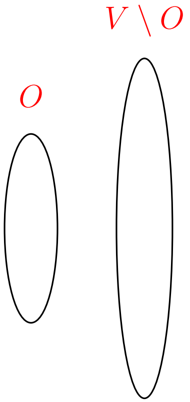

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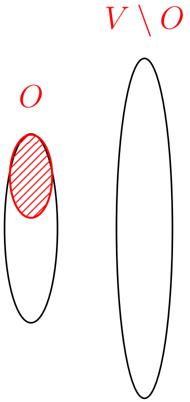

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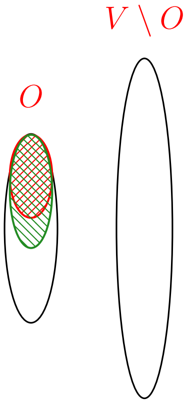

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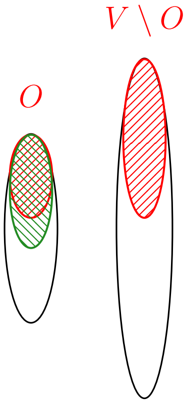

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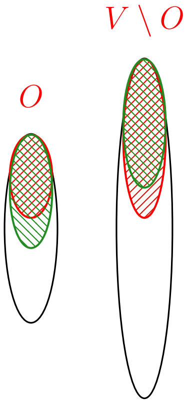

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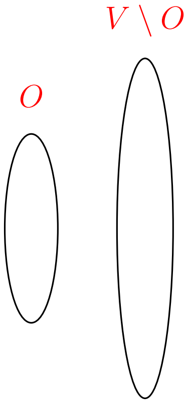

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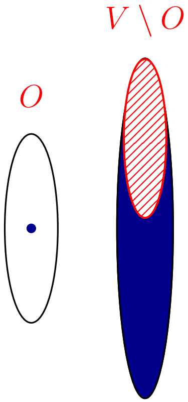

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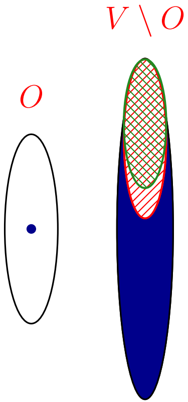

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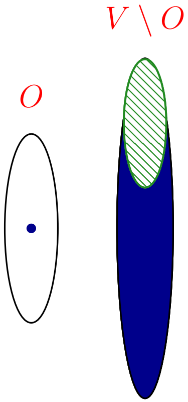

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  - In particular, for obtaining a maximal matching?

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