Tight Space-Approximation Tradeoff for the Multi-Pass Streaming Set Cover Problem

Sepehr Assadi

University of Pennsylvania

The Set Cover Problem

- Input: A collection of m sets $S_1, \ldots, S_m$ from a universe $[n]$.

- Goal: Choose a smallest subset C of the sets from $S_1, \ldots, S_m$ such that C covers $[n]$, i.e., $\bigcup_{i \in C} S_i = [n]$.

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We use OPT to denote the optimal solution size.

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- Operation research, machine learning, web host analysis, ...

The Set Cover Problem: Classical Setting

Theoretical aspects:

- One of Karp's original 21 NP-hard problems [Karp, 1972].
- The greedy algorithm that picks the "best" set in each iteration achieves $\ln(n)$ approximation [Johnson, 1974, Slavík, 1997].
- No better approximation factor is possible in polynomial time unless $\mathsf{P} = \mathsf{NP}$ [Lund and Yannakakis, 1994, Feige, 1998, Dinur and Steurer, 2014, Moshkovitz, 2015].

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- The greedy algorithm is highly efficient and surprisingly accurate.
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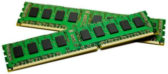
as long as the dataset is relatively small!

The Set Cover Problem: Big Data Scenario

[Cormode et al., 2010]: A direct implementation of the greedy algorithm scales surprisingly poorly when the data size grows.



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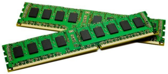

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One approach: the streaming model for the set cover problem introduced by [Saha and Getoor, 2009].

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Small space:

1. Semi-streaming space, i.e., $\widetilde{O}(n)$.
2. Sub-linear space, i.e., $o(mn)$.

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- For theoretical purposes: understanding the space complexity of streaming algorithms in absence of time complexity restrictions.
- For practical purposes: we rarely need the full power of such exponential time computation anyway.

State of the Art

Many interesting results: [Saha and Getoor, 2009, Cormode et al., 2010, Emek and Rosén, 2014, Demaine et al., 2014, Badanidiyuru et al., 2014, Indyk et al., 2015, Har-Peled et al., 2016, Chakrabarti and Wirth, 2016, Assadi et al., 2016, McGregor and Vu, 2016, Bateni et al., 2016].

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Short summary: to ensure efficiency, we need more than $\widetilde{O}(n)$ space and more than one pass!

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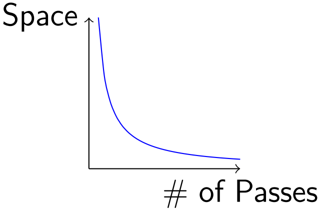

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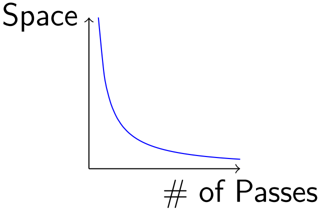

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[Har-Peled et al., 2016]:

Conjecture. This tradeoff is tight for small approximation factors.

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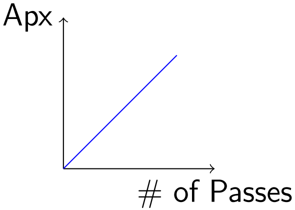

the pass-approximation tradeoff

Not a typical pass-approximation tradeoff!

Motivating Questions

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  Answer: The above space-approximation tradeoff is essentially tight even allowing $\mathrm{polylog}(n)$ passes over the stream!

Our Main Result

Theorem

For $\alpha = o(\log n)$, any p-pass α-approximation algorithm (deterministic or randomized) for the streaming set cover requires $\widetilde{\Omega}\left(\frac{1}{p} \cdot mn^{1/\alpha}\right)$ space, even if the sets are arriving in a random order.

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Remark.

- The lower bound has nothing to do with the NP-hardness of approximating set cover!
- It holds in the regime when $\mathrm{OPT} = O(1)$ in which case set cover admits a trivial poly-time algorithm in the classical setting.

Further Results

- We show that with proper modifications, the algorithm of [Har-Peled et al., 2016] can be implemented in $\widetilde{O}(mn^{1/\alpha})$ space, matching our lower bound up to logarithmic factors.

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- Using similar ideas, we can also prove a tight lower bound for the space complexity of $(1-\varepsilon)$-approximating the streaming maximum coverage problem.

Communication Complexity

We use communication complexity to prove our lower bound.

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Two-player Communication Model:

- Alice gets the sets $S_1, \ldots, S_m$ and Bob gets $T_1, \ldots, T_m$.

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- Alice and Bob are allowed to communicate with each other to compute the set cover.
- Communication Complexity $\mathrm{CC}(\textsc{SetCover})$: minimum amount of communication needed to solve the problem w.p. $\geq 2/3$.

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Hence, space complexity of p-pass streaming algorithms for the set cover problem $\geq \frac{1}{p} \cdot \mathrm{CC}(\textsc{SetCover})$.

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Fix an $\alpha \ll \log n$. Define:

$\textsc{SetCover}$: the two-player communication problem of finding an α-approximation to the set cover problem.

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2. Each instance sampled from $\mathcal{D}^{\mathsf{N}}$ (No instance), has $\mathsf{OPT} > 2\alpha$ w.p. $1 - o(1)$.
3. Distinguishing between Yes and No instances requires $\widetilde{\Omega}(mn^{1/\alpha})$ communication.

A Hard Input Distribution for SetCover

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   - Each element $e \in Z_i$ goes to $\begin{cases} S_i & \text{w.p. } 1/3 \\ T_i & \text{w.p. } 1/3 \\ \text{both } S_i \text{ and } T_i & \text{o.w.} \end{cases}$.

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To create a Yes instance, we choose $i^\star \in [m]$ uniformly at random and let $Z_i = [n]$.

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- $S_i \cup T_i = Z_i$, hence covering everything except for $n^{1-1/\alpha}$ elements.
- $S_i \cup T_j$ covers $\approx 8n/9$ elements as S_i and T_j are two independent random sets of size $\approx 2n/3$.

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- The probability that a fixed element $e \in [n]$ is not covered by a fixed α-subset is: $\approx \left(1/n^{1/\alpha}\right)^{\alpha} \approx \frac{1}{n}$

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- The probability that a fixed element $e \in [n]$ is not covered by a fixed α-subset is: $\approx \left(1/n^{1/\alpha}\right)^{\alpha} \approx \frac{1}{n}$
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A Hard Input Distribution for SetCover

OPT in Yes instances? 2; pick $S_{i^\star}$ and $T_{i^\star}$ as $S_{i^\star} \cup T_{i^\star} = Z_{i^\star} = [n]$.

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- Use some concentration result + union bound to finalize the claim.

The Lower Bound for Set Cover on $\mathcal{D}$

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- Intuitively, to "catch" any of the missing elements, Alice and Bob need to communicate $\Omega(n^{1/\alpha})$ elements.
- Can be formalized using a reduction from the set disjointness problem.

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- Can be formalized using information complexity and a direct sum-style argument.
- There are some subtle technical challenges in applying this idea!

The Lower Bound for SetCover: Wrapup

Distinguishing between Yes and No instances of $\mathcal{D}$ requires

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Some additional steps are required to extend this lower bound to random order streams.

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For the multi-pass streaming set cover problem:

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- We know it cannot be one pass [Assadi et al., 2016].
- Conjectured by [Har-Peled et al., 2016] that $\Theta(\alpha)$ is tight.

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