Towards a Unified Theory of Sparsification for Matching Problems

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Joint work with Aaron Bernstein (Rutgers)

Sparsifiers

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Canonical examples:

- Cut sparsifiers: preserve cut-value between bi-partitions [Karger, 1994, Benczúr and Karger, 1996, Fung et al., 2011];
- Spectral sparsifiers: preserve Laplacian spectrum of the graph [Spielman and Teng, 2011, Batson et al., 2009];
- Spanners: preserve pairwise distances [Awerbuch, 1985, Peleg and Schäffer, 1989];
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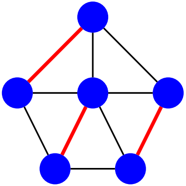
This talk: Are there efficient matching sparsifiers?

Matching

- Matching: A collection of vertex-disjoint edges.



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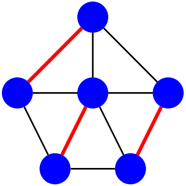

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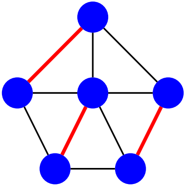

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$\mu(G)$: size of a maximum matching in G.

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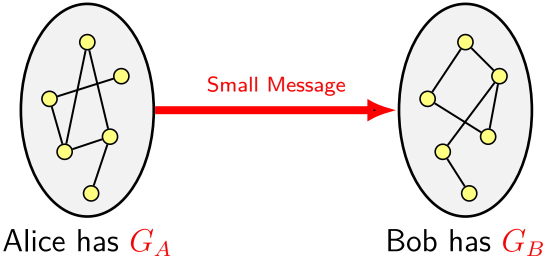
Let us instead consider examples of what we expect from a "good" matching sparsifier in the context of known matching problems.

One-Way Communication Complexity of Matching

Consider the following problem:

- Alice and Bob are given graphs $G_A(V, E_A)$ and $G_B(V, E_B)$.
- Alice wants to send a single message to Bob so Bob can compute a maximum matching of $G_A \cup G_B$.
- What is the tradeoff between length of the message and approximation ratio?



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Studied by [Goel et al., 2012, Lee and Singla, 2017] owing to its close connection to streaming and online batch-arrival algorithms for maximum matching.

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For original maximum matching problem, we would like Alice to be able to send a matching sparsifier!

Other Examples

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Compute a sparse subgraph H of G such that random subgraphs of H have a large matching compared to random subgraphs of G.

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Fault tolerant matching problem

Fault-tolerant subgraphs studied extensively for spanners and distance preservers. [Chechik et al., 2009, Peleg, 2009, Baswana et al., 2016, Bodwin et al., 2017, Bodwin et al., 2018] $\cdots$

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Very recently also used to design randomized composable coresets for matching [Assadi et al., 2019].

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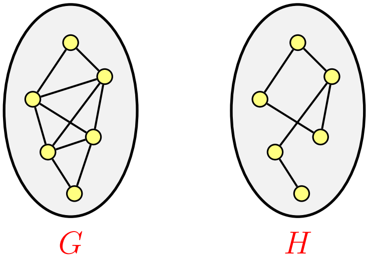

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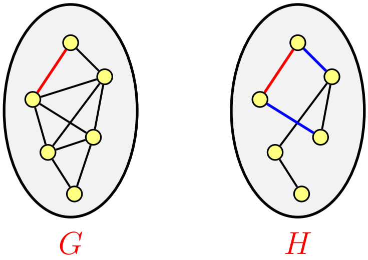

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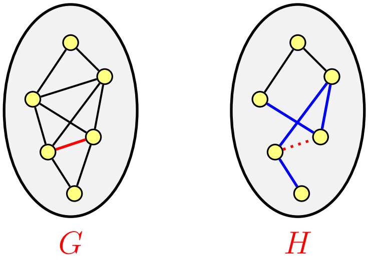

EDCS as a Matching Sparsifier

Basic properties:

- A (β, ε)-EDCS has $O(n\beta)$ edges.
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[This work]:

> *An EDCS can act as a robust matching sparsifier under different notions of sparsification.*

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f-fault-tolerant matching problem:
$(1.5+\varepsilon)$-approximation with $O_\varepsilon(n+f)$ edges.

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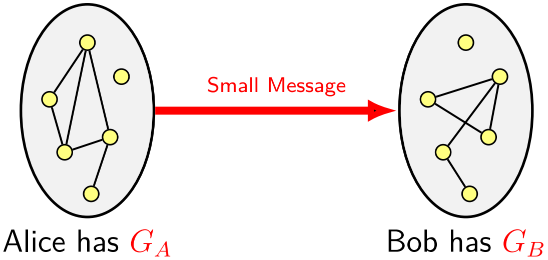

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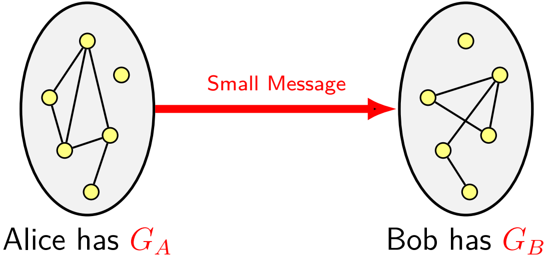
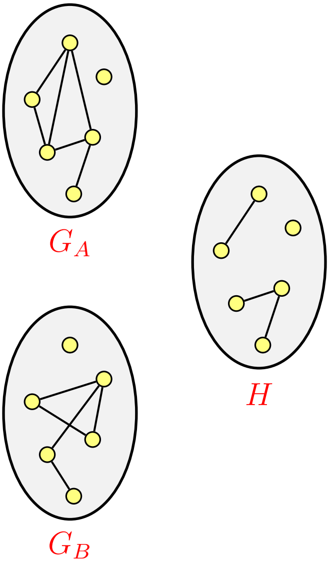


Our solution: Alice sends a $(1/\varepsilon^2, \varepsilon)$-EDCS H of G_A to Bob.

Proof of Correctness



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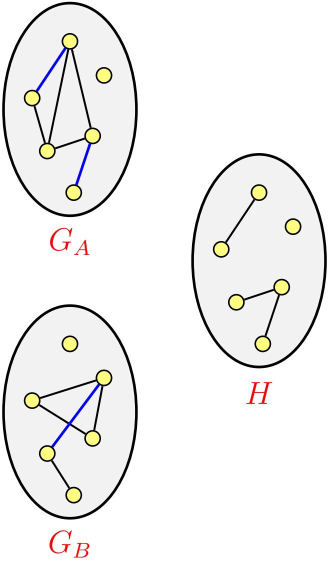

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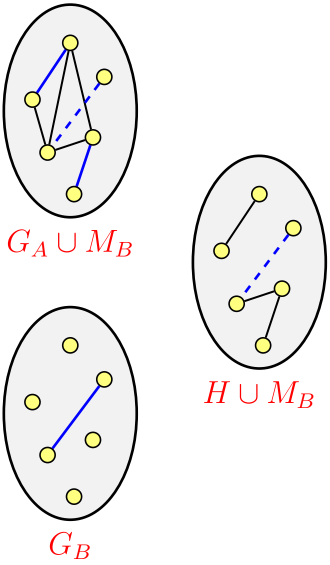
- Fix a maximum matching $M_A \cup M_B$ of $G_A \cup G_B$.
- $H \cup M_B$ contains a $(\beta + 2, 2\varepsilon)$-EDCS of $G_A \cup M_B$:



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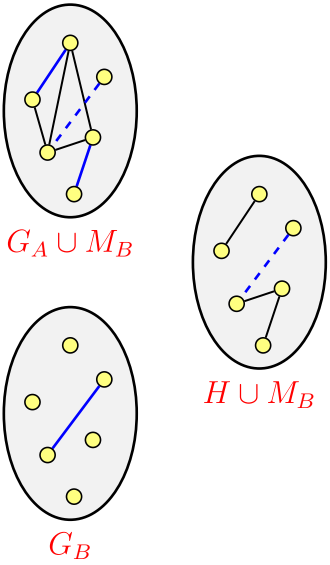

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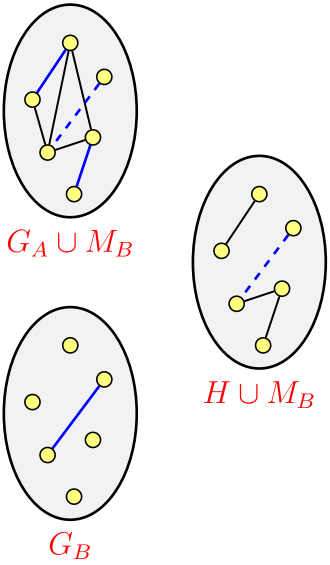

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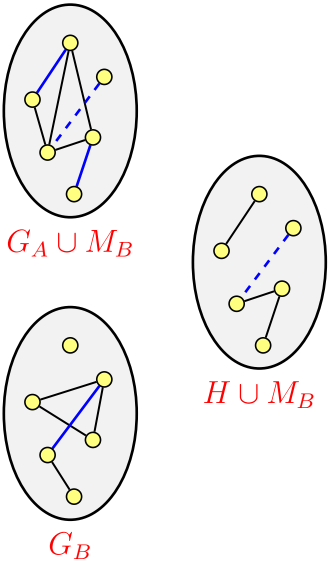
- $H \cup M_B \subseteq H \cup G_B$ known to Bob.



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EDCS Contains a Large Matching

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Part two: Reduce the general case to bipartite graphs.

- Uses robustness properties of EDCS that we prove in this paper with a simple application of Lovasz Local Lemma.
- Entirely different from and significantly simpler than [Bernstein and Stein, 2016].

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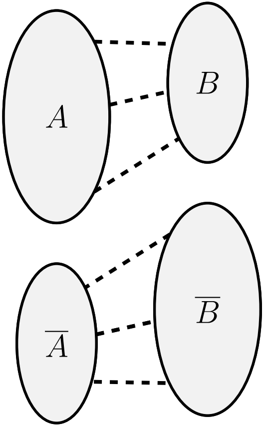
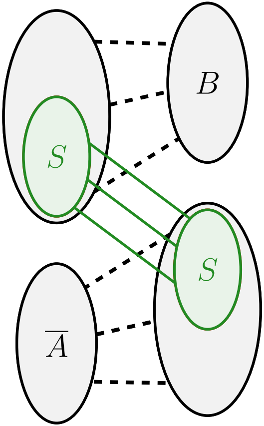


EDCS H

Bipartite Graphs

- H is a (β, ε)-EDCS of G with maximum matching size $\mu(H)$.
- Hall's theorem: No A to $\overline{B}$ edge; $\left|\overline{A} \cup B\right| = \mu(H)$.
- G has a matching of size $\mu(G) \geq \mu(H)$: $|S| = 2(\mu(G) - \mu(H))$.



Matching M in G.

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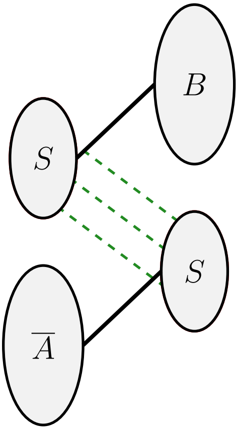

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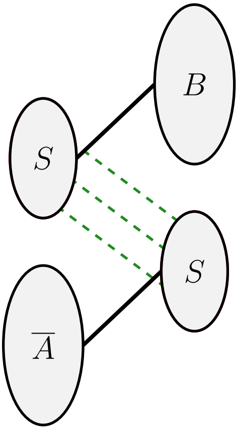

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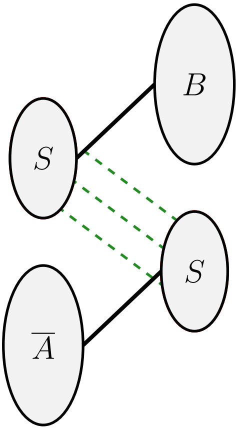

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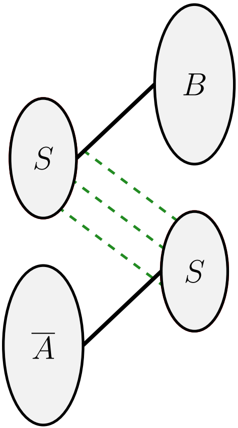

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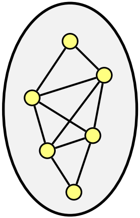

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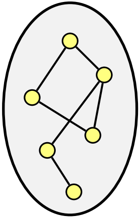

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- Fix a (β, ε)-EDCS H of G.
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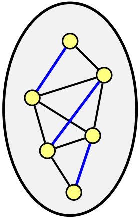

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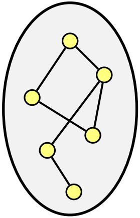

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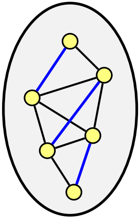

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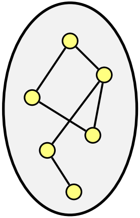

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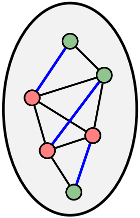

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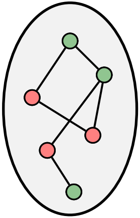

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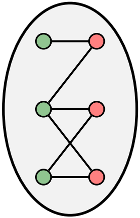
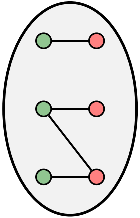


G'



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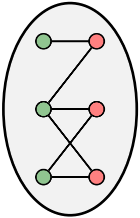

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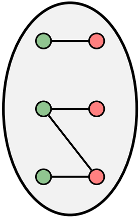

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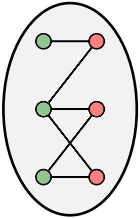

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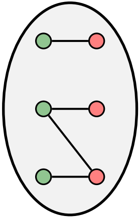

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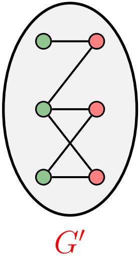

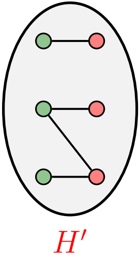

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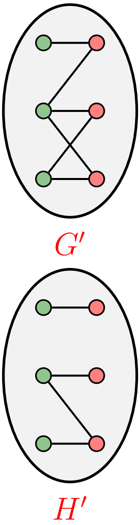
- $(1.5 + \varepsilon)$-approximation by the result on bipartite graphs.



Concluding Remarks

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One-way communication complexity of matching:
$(1.5 + \varepsilon)$-approximation with $O_\varepsilon(n)$ communication.

Stochastic matching problem:
$(1.5 + \varepsilon)$-approximation with max-degree $O_\varepsilon\left(\frac{\log(1/p)}{p}\right)$.

f-fault-tolerant matching problem:
$(1.5 + \varepsilon)$-approximation with $O_\varepsilon(n + f)$ edges.

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Thank you!

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