Matching Size and Matrix Rank Estimation in Data Streams

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Joint work with Sanjeev Khanna (Penn), and Yang Li (Penn)

The Streaming Model

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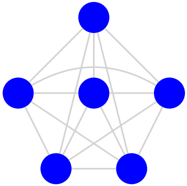
- Input is presented as a data stream, for instance, as a sequence of edges in case of a graph input.
- Algorithm sees the entire input once and only has a small space to store information about the input as it passes by.
- At the end of the sequence, the algorithm outputs a solution using only the stored information.

The Streaming Model: Example

A graph stream:

Stream:



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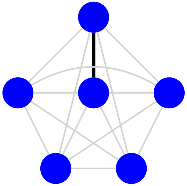

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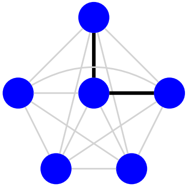
A graph stream:

Stream: $+e_1, +e_7$



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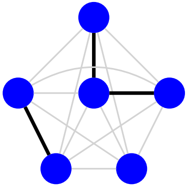

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A graph stream:

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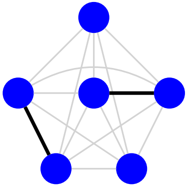

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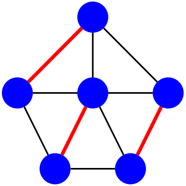
- Dynamic Streams. Contains both positive and negative updates.

Matchings in Graphs

- Matching: A collection of vertex-disjoint edges.



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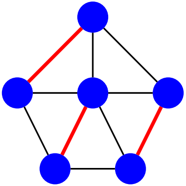

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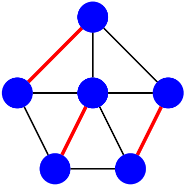

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Maximum Matching problem: Find a matching with a largest number of edges.

Matchings in Graphs

Maximum matching is a fundamental problem with many applications.

- Many celebrated algorithms in the classical setting: Ford-Fulkerson, Edmond's, Hopcroft-Karp, Mucha-Sankowski, Madry's, ...
- Studied in various computational models: distributed, parallel, online, sub-linear time, streaming, ...

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This talk: sublinear space algorithms for the matching problem in the streaming model.

Finding a Matching vs Estimating Size

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Are there any qualitative difference in the space needed to achieve these goals?

Finding Large Matchings

Arguably the most studied problem in the graph streaming literature.

[McGregor, 2005] [Feigenbaum et al., 2005] [Eggert et al., 2009] [Epstein et al., 2011] [Goel et al., 2012] [Konrad et al., 2012] [Zelke, 2012] [Ahn et al., 2012] [Ahn and Guha, 2013] [Guruswami and Onak, 2013] [Kapralov, 2013] [Kapralov et al., 2014] [Crouch and Stubbs, 2014] [McGregor, 2014] [Chitnis et al., 2015] [Ahn and Guha, 2015] [Esfandiari et al., 2015] [Konrad, 2015] [Bury and Schwiegelshohn, 2015] [Assadi et al., 2016] [Chitnis et al., 2016] [McGregor and Vorotnikova, 2016b] [Esfandiari et al., 2016] [Paz and Schwartzman, 2017] [Ghaffari, 2017] [Kale et al., 2017] ...

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Dynamic streams:

- $\Omega(n^2/\alpha^3)$ space is necessary for α-approximation [Assadi et al., 2016].
- $\widetilde{O}(n^2/\alpha^3)$ space is sufficient for α-approximation [Assadi et al., 2016, Chitnis et al., 2016].

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- Lower bounds (Insertion-only streams):
  - $(1+\varepsilon)$-approximation requires $\Omega(n^{1-O(\varepsilon)})$ space [Esfandiari et al., 2015, Bury and Schwiegelshohn, 2015].
  - Deterministic α-approximation requires $\Omega(n/\alpha)$ space [Chakrabarti and Kale, 2016].

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- $\widetilde{O}(n/\alpha^2)$ *space in insertion-only streams.*
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In constrast, to find an α-approximate matching, the space necessary is:

- $\Omega(n/\alpha)$ in insertion-only streams.
- $\Omega(n^2/\alpha^3)$ in dynamic streams.

Algorithms for α-Estimation of Matching Size

The main ingredient of our algorithms is the following sampling lemma:

Lemma (Vertex Sampling Lemma)

Let H be a subgraph of G obtained by sampling each vertex independently w.p. $1/\alpha$. Define:

μ_G*: the maximum matching size in* G,

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Then, w.h.p.,

$$\frac{\mu_G}{\alpha^2} \leq \mu_H \leq \frac{2\mu_G}{\alpha}.$$

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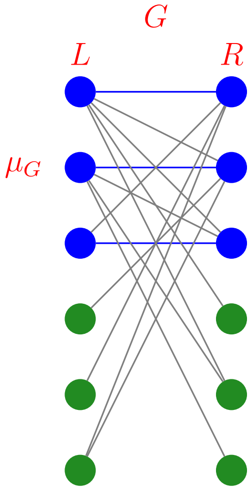
Therefore, maximum matching size in H is an α-estimation for the maximum matching size in G.

Proof by Picture

Any graph G with a maximum matching size of μ_G looks as follows:

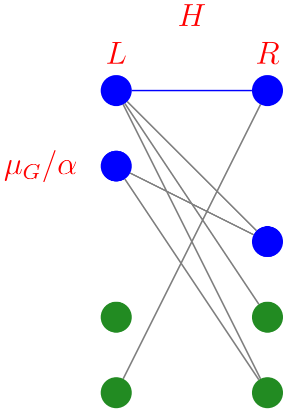
- A matching of size μ_G between the blue vertices.
- No edges between the green vertices.



Proof by Picture

The vertex sampled graph H then look as follows:

- A matching of size μ_G/α^2 between the blue vertices $\implies \mu_H \geq \mu_G/\alpha^2$.
- All edges are incident on μ_G/α blue vertices $\implies \mu_H \leq 2\mu_G/\alpha$.



An α-Estimation Algorithm

To distinguish between graphs with maximum matching of size $\geq k$ and $o(k/\alpha)$:

1. Sample each vertex in G w.p. $1/\alpha$ to obtain H.
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Can be implemented in:

- $\widetilde{O}(k/\alpha^2) = \widetilde{O}(n/\alpha^2)$ in insertion-only streams.
- $\widetilde{O}(k^2/\alpha^4) = \widetilde{O}(n^2/\alpha^4)$ in dynamic streams.

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We make progress on each of these questions.

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Near-optimal approximation of maximum matching size may require almost quadratic space even in insertion-only streams.

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Any randomized $(1+\varepsilon)$-approximate estimation of maximum matching size requires:

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$\mathsf{RS}(n)$ denotes the maximum number of edge-disjoint induced matchings of size $\Theta(n)$ in an n-vertex graph:

[Fischer et al., 2002]$n^{\Omega(1/\log\log n)} \leq \mathsf{RS}(n) \leq n/\log n$[Fox et al., 2015]

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Furthermore, even if we restrict to sparse graphs with arboricity $O(\alpha)$, $\Omega(\sqrt{n}/\alpha^{2.5})$ space is necessary.

There is an active line of research on estimating matching size of bounded arboricity graphs in graph streams [Chitnis et al., 2016] [Bury and Schwiegelshohn, 2015] [Esfandiari et al., 2015] [McGregor and Vorotnikova, 2016b] [Cormode et al., 2016] [McGregor and Vorotnikova, 2016a] . . .

Schatten p-Norms

Given an $n \times n$ matrix A, for any $p \in [0, \infty)$:

Schatten p-norm of A is the p-th frequency moment of vector of singular values $(\sigma_1, \ldots, \sigma_n)$ of A.

$$\|A\|_p := \left(\sum_{i=1}^{n} \sigma_i^p\right)^{1/p}$$

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- $\|A\|_0 =$ Rank of A.
- $\|A\|_1 =$ Trace norm of A.
- $\|A\|_2 =$ Frobenius norm of A.
- $\|A\|_\infty =$ Operator norm of A.

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Previous work:

- For $p = 0$, $\Omega(n^{1-g(\varepsilon)})$ space is necessary [Bury and Schwiegelshohn, 2015].
- For $p \in (0, \infty) \setminus 2\mathbb{Z}$, $\Omega(n^{1-g(\varepsilon)})$ space is necessary [Li and Woodruff, 2016].
- For $p \in 2\mathbb{Z} \setminus \{0\}$, $\Omega(n^{1-2/p})$ space is necessary [Li and Woodruff, 2016] (and is sufficient for sparse matrices).

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We answer this question for the case of rank computation.

Matrix Rank Computation in Data Streams

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As a corollary, all our lower bounds for matching size estimation also extend to the matrix rank computation problem. In particular,

- An $\Omega(n^{2-O(\varepsilon)})$ space lower bound for $(1+\varepsilon)$-estimation of rank in dense matrices.
- An $\widetilde{\Omega}(\sqrt{n})$ space lower bound for any $\mathrm{polylog}(n)$-estimation of rank in sparse matrices.

An $\Omega(n^{2-O(\varepsilon)})$ Lower Bound for Dynamic Streams

Theorem

Any randomized $(1+\varepsilon)$-approximate estimation of maximum matching size requires $\Omega(n^{2-O(\varepsilon)})$ space in dynamic streams.

Previous Approaches: An $n^{1-O(\varepsilon)}$ Lower Bound

Consider the following two-player one-way communication problem.

MaxMatching:

1. Alice is given a matching M on vertices V.
2. Bob is given a collection of edges E_B on vertices V.
3. Alice sends a single message to Bob and Bob outputs an estimation of maximum matching size in $G(V, M \cup E_B)$.

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Fact. CC(MaxMatching) $\leq$ space complexity of any streaming algorithm for estimating maximum matching size.

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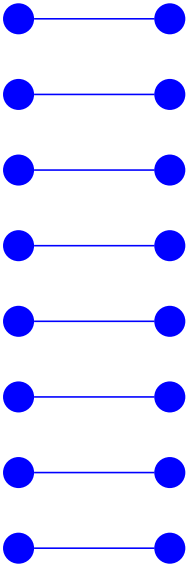
- Alice is given a random subset of size $n/2$ from a fixed perfect matching between L and R.



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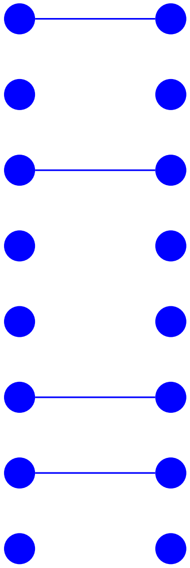

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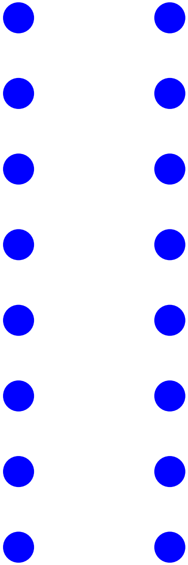

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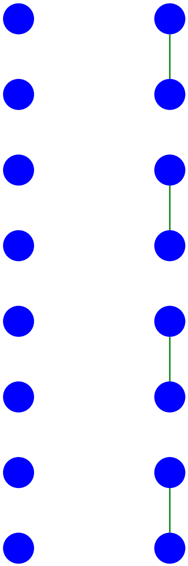

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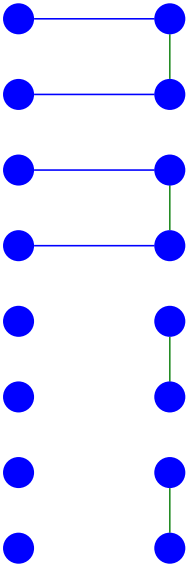
- Yes case: Each edge of Bob's matching is incident on even number of Alice's matching $\implies$ $\textsc{MaxMatching} = 3n/4$.



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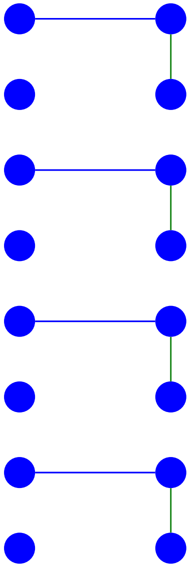
- Bob is given a matching of size $n/2$ incident on R.
- No case: Each edge of Bob's matching is incident on odd number of Alice's matching $\implies$ $\textsc{MaxMatching} = n/2$.



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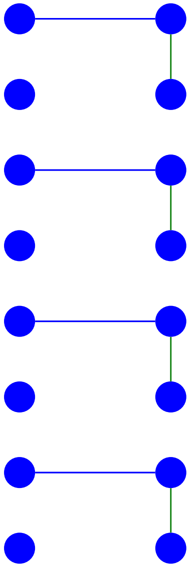
- A better than $3/2$-approximation distinguishes between the two cases.
- Distinguishing between the two cases requires $\Omega(\sqrt{n})$ communication by a reduction from the boolean hidden matching problem of [Gavinsky et al., 2007].



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The hope is that communication complexity of this problem is now $\geq t \cdot \text{CC}(\text{MaxMatching})$.

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There are three main obstacles in implementing this idea:

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Formalizing the lower bound $\implies$ a direct-sum style argument using information complexity.

Ruzsa-Szemerédi Graphs

Definition ((r,t)-RS graphs)

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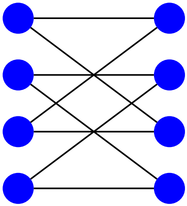

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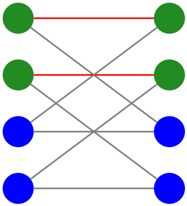

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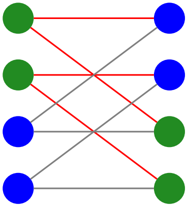

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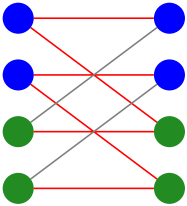

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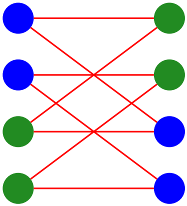

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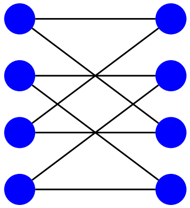
A Simple Lower Bound for Insertion-Only Streams

- Let G_1 be an (r, t)-RS bipartite graph on n vertices on each side.



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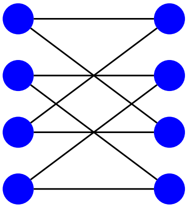

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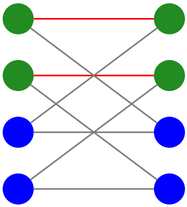

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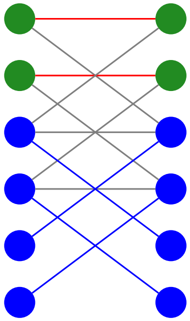

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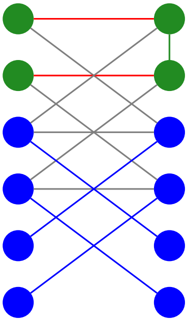

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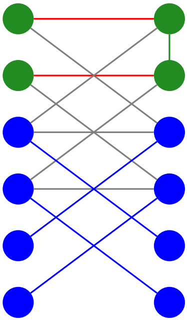

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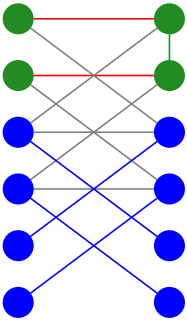

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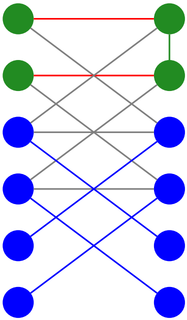
- To solve this for an unknown matching $M_{j^\star}$, the message length must be $\geq t \cdot \text{CC}(\textsc{MaxMatching})$.



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Main limitation of this approach:

- Requires $r = \Theta(n) \implies t = \mathsf{RS}(n)$.
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We bypass the $r = \Theta(n)$ limitation in dynamic streams using the characterization result of [Li et al., 2014, Ai et al., 2016] in terms of the simultaneous communication complexity.

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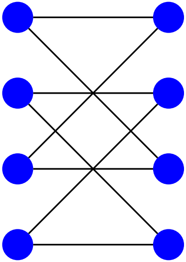
[Ai et al., 2016]: Communication lower bounds in this model imply identical space lower bound for dynamic streaming algorithms.

A Hard Input Distribution

- Each player is given an (r, t)-RS graph on N vertices.



Local view of P^i

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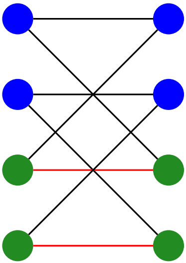

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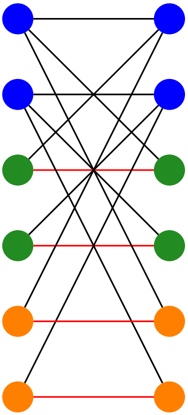

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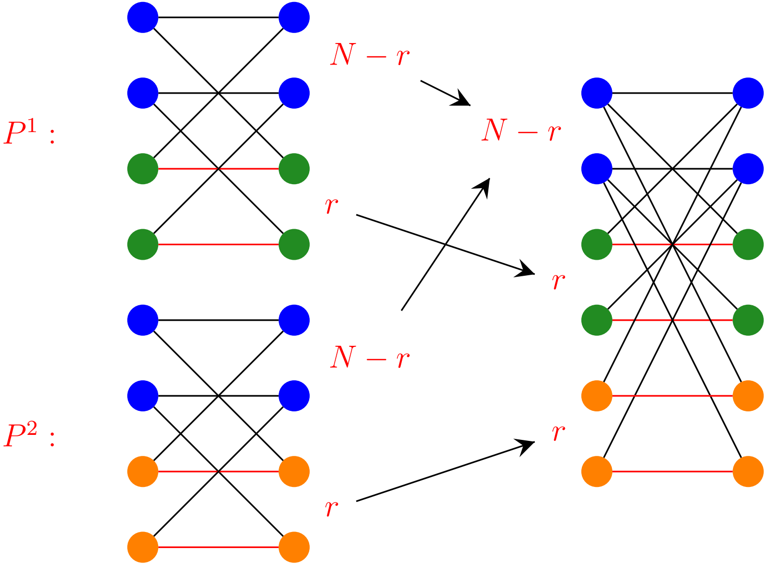

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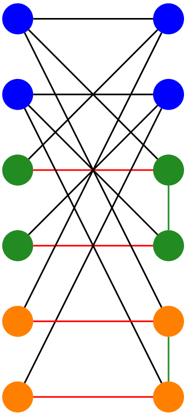
- To the referee, we provide k subgraphs $E^{(1)}_B, \ldots, E^{(k)}_B$ such that each pair $(M^{(i)}_{j^\star}, E^{(i)}_B)$ forms the same instance of MaxMatching.



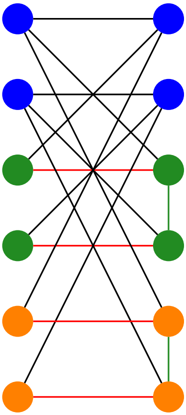
Global view

Proof Sketch

- Define $\textsc{MaxMatching}(M, E_B) := \textsc{MaxMatching}(M^{(1)}_{j^\star}, E^{(1)}_B) = \ldots = \textsc{MaxMatching}(M^{(k)}_{j^\star}, E^{(k)}_B)$.
- The maximum matching size in G is:

  $$\approx 2(N - r) + k \cdot \textsc{MaxMatching}(M, E_B)$$

- For $r = N^{1-o(1)}$ and $k \approx \frac{1}{\varepsilon} \cdot N^{o(1)}$, $\textsc{MaxMatching}(M, E_B)$ is the dominating term for $(1 + \varepsilon)$-approximation.



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- To conclude,

$$N^{1-O(\varepsilon)} \leq k \cdot s/t \implies s \geq \frac{1}{k} \cdot t \cdot N^{1-O(\varepsilon)} \approx N^{2-O(\varepsilon)}$$

as $t = N^{1+o(1)}$ by [Alon et al., 2012] and $k = \Theta_\varepsilon(N^{o(1)})$.

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Theorem

Any randomized $(1+\varepsilon)$-approximate estimation of maximum matching size requires $\Omega(n^{2-O(\varepsilon)})$ space in dynamic streams.

Concluding Remarks

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- Similar-in-spirit lower bounds for Schatten p-norms for $p > 0$?

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