CS 860: Modern Topics in Graph Algorithms **University of Waterloo: Winter 2024**

Lecture 3: $(\Delta + 1)$ Vertex Coloring

January 29, 2023

Instructor: Sepehr Assadi

Topics of this Lecture

1 The $(\Delta + 1)$ Vertex Coloring Problem

A proper c-coloring of an undirected graph is any assignment of colors from the palette $\{1, 2, \ldots, c\}$ to the vertices of the graph such that no two adjacent vertices receive the same color. Graph coloring is another highly fundamental problem in TCS and graph theory with a wide range of applications. The following is one of the most basic and applicable forms of graph coloring.

Problem 1 (**(Δ+1) Vertex Coloring**)**.** The $(\Delta+1)$ vertex coloring problem is defined as the problem of finding a $(\Delta + 1)$-coloring of a given graph with maximum degree Δ.

Every graph admits a $(\Delta+1)$ coloring[1]: we can color the vertices greedily and by the pigeonhole principle we never run out of color for each vertex as number of its neighbors is less than the available colors[2]. The runtime of this algorithm is $O(m+n)$ time on n-vertex m-edge graphs which is clearly "optimal" as we need this much time simply to read the input. Or is this optimal?

It turns out that we can in fact find a $(\Delta + 1)$ coloring of a graph even faster than reading the entire input! Such an algorithm is called a *sublinear-time* algorithm as its runtime is sublinear in its input size.

[1] And this is tight for cliques and odd-cycles and in fact only for these graphs: by Brook's theorem, any connected graph other than a clique or an odd-cycle admits a Δ-coloring.

[2] In fact, $(\Delta + 1)$ coloring has an "ultra greedy" property: any partial solution, i.e., a partial coloring of vertices, can be extended to a proper coloring without modification.

In this lecture, we cover an algorithm due to Assadi, Chen, and Khanna [ACK19] that solves this problem, using randomization, in $\widetilde{O}(n^{3/2})$ time (recall that $\widetilde{O}(f) = O(f \cdot \text{poly}\log(f))$, i.e., it suppresses logarithmic factors as well). Formally,

Theorem 1 ([ACK19])**.** *There is a randomized algorithm that given access to* both *adjacency list and adjacency matrix of the input graph $G = (V, E)$ with maximum degree Δ, with high probability outputs a $(\Delta+1)$ coloring of G in $\widetilde{O}(n^{3/2})$ time and otherwise outputs FAIL. The algorithm does not need a prior knowledge of Δ.*

We will mainly focus on proving a weaker version of this theorem in this lecture: the algorithm we design will *read* (a.k.a *query*) only $\widetilde{O}(n^{3/2})$ entries from the input (but we will not fully analyze its running time), and it also assumes the prior knowledge of Δ (unlike the first one, this assumption is quite easy to remove and we will provide a hint for doing so at the end of the lecture).

2 Palette Sparsification Theorem

The key to proving Theorem 1 is the following purely combinatorial result established in [ACK19].

Theorem 2 (**Palette Sparsification Theorem** [ACK19])**.** *Let $G = (V, E)$ be any graph with n vertices and maximum degree Δ. Suppose for every vertex $v \in V$, we sample a list $L(v)$ of $s = \Theta(\log n)$ colors independently and uniformly at random from $\{1, \ldots, \Delta+1\}$ (with repetition). Then, with high probability, G can be colored by coloring every $v \in V$ from the list $L(v)$; in other words, G is list-colorable from $\{L(v) \mid v \in V\}$.*

The term 'sparsification' in the statement of this theorem refers to *two* different properties: firstly and more obviously, Theorem 2 shows that instead of working with the same palette of size $\Delta+1$ for every vertex in the original problem, we can *sparsify* the palettes across vertices to each have only $O(\log n)$ size (although different across vertices); the second and perhaps less obvious property is that after performing sampling of the colors (which does not require us to even "look" at the graph), we can *sparsify* the edges of the graph dramatically also. This is because we now only need to consider edges whose sampled lists of the two endpoints coincide (any other edge cannot create any conflict in the list-coloring step as the lists of its endpoints are disjoint – that means we can safely ignore all such edges). See Figure 1 for an illustration. The following lemma bounds the number of remaining edges.

Lemma 3. *Let $G = (V, E)$ be any graph with maximum degree Δ and lists $\{L(v) \mid v \in V\}$ be as in Theorem 2. Let $H = (V, E_H)$ be a subgraph of G obtained by picking only edges (u, v) such that $L(u) \cap L(v) \neq \emptyset$. We refer to H as the* ***conflict graph****. Then, with high probability, the maximum degree of H is $O(\log^2(n))$ and thus H has $O(n \log^2(n))$ edges.*

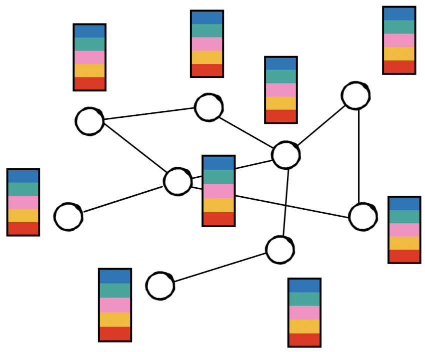
Before getting to prove Lemma 3, we need a quick reminder on the standard Chernoff bound.

Proposition 4 (**Chernoff bound**)**.** *Suppose $X_1, \ldots, X_n$ are* independent *random variables in $[0, 1]$ and $X = \sum_i X_i$. Then, for any $\delta > 0$,*

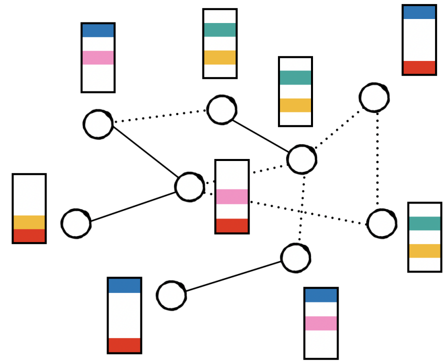
$$\Pr\left(|X - \mathbb{E}[X]| \geqslant \delta \cdot \mathbb{E}[X]\right) \leqslant 2 \cdot \exp\left(-\frac{\delta^2 \cdot \mathbb{E}[X]}{(2+\delta)}\right).$$

Proof of Lemma 3. Recall that s is the size of the lists sampled in Theorem 2. Fix any vertex $v \in V$ and its list of colors $L(v)$ and for any neighbor u of v in G, let $X_u \in \{0, 1\}$ be an indicator random variable which is 1 iff v is neighbor to u in H. For u to be neighbor of v, $L(u) \cap L(v)$ should be non-empty. Thus,

$$\Pr(X_u = 1) \leqslant \sum_{c \in L(v)} \Pr(c \in L(u)) = \frac{s^2}{\Delta+1},$$



(a) A graph $G = (V, E)$ with its original palettes of $(\Delta + 1)$ colors.



(b) The sampled palettes of the vertices and the resulting conflict graph – the dotted edges can be ignored now.

Figure 1: An illustration of the Palette Sparsification Theorem.

where the first inequality is by union bound and the equality is because $|L(v)| = |L(u)| = s$ and the choices of $L(v)$ and $L(u)$ are independent. Moreover, $\deg_H(v) = \sum_{u \in N_G(v)} X_u$ and thus

$$\mathbb{E}\left[\deg_H(v)\right] \leqslant \Delta \cdot \frac{s^2}{(\Delta+1)} = O(\log^2(n)),$$

by the choice of $s = \Theta(\log(n))$.

As $\deg_H(v)$ is sum of independent[3] 0/1-random variables X_u for $u \in N_G(v)$, we can apply Chernoff bound (Proposition 4 with $\delta = 10$) and obtain that

$$\Pr\left(\deg_H(v) \geqslant 10 \cdot \mathbb{E}\left[\deg_H(v)\right]\right) \leqslant 2 \cdot \exp\left(-\frac{100\log^2(n)}{12}\right) < \frac{1}{n^8}.$$

A union bound over all n vertices now finalizes the proof. □

2.1 A Sublinear-Query Algorithm from Palette Sparsification

Before getting to discuss the proof of the Palette Sparsification Theorem, let us now showcase its power by designing a sublinear-query (although still not time) algorithm from it.

Algorithm 1. A randomized $\widetilde{O}(n^2/\Delta)$-query algorithm for $(\Delta + 1)$ coloring.

(i) For every vertex $v \in V$, sample a set $L(v)$ of $s = \Theta(\log n)$ colors (for the same hidden constant as in Theorem 2) from $\{1, \ldots, \Delta + 1\}$ uniformly at random and independently.

(ii) For $c \in [\Delta + 1]$, let V_c be the vertices $v \in V$ which have sampled the color c in their lists $L(v)$.

(iii) For every $c \in [\Delta + 1]$ and $u, v \in V_c$, query the adjacency matrix of G to determine if (u, v) is an edge in G or not; if it is an edge, add (u, v) to the conflict graph H.

(iv) List-color H from the sampled lists $\{L(v) \mid v \in V\}$ and output this coloring as the final answer; if no such coloring is found, output FAIL.

[3] X_u's are independent as we already conditioned on $L(v)$, and after that, $L(u)$'s for $u \in N_G(v)$ are chosen independently.

Lemma 5. *With high probability, Algorithm 1 reads at most $O(\frac{n^2}{\Delta} \cdot \log^2(n))$ entries of the adjacency matrix of G and outputs a proper $(\Delta+1)$ coloring of G.*

Proof. Fix a color $c \in [\Delta+1]$. In the algorithm (and Theorem 2), there are $n \cdot s$ samples of colors in total and each sample is c with probability $1/(\Delta+1)$. Thus, the expected number of vertices that sample a color c is at most $(n \cdot s)/(\Delta+1)$. Given all these choices are also done independently, a simple application of Chernoff bound (Proposition 4 with $\delta = 12$) implies that

$$\Pr\left(|V_c| \geqslant 12 \cdot \mathbb{E}\,|C|\right) \leqslant \exp\left(-\frac{144 \cdot n \cdot s}{14 \cdot (\Delta+1)}\right) \ll n^{-5},$$

where we used the fact that $\Delta \leqslant n$ and $s \geqslant \log n$.

A union bound over all choices of c now implies that with high probability, for every $c \in [\Delta+1]$, we have $|V_c| = O(n \log n/\Delta)$. Thus, the total number of queries to the adjacency matrix is

$$(\Delta+1) \cdot O(1) \cdot \left(\frac{n \log n}{\Delta}\right)^2 = O(\frac{n^2 \log^2 n}{\Delta}).$$

This bounds the number of queries of the algorithm (with high probability) as desired.

Finally, by Theorem 2, the subgraph H is also list-colorable from the sampled lists with high probability; given that every list is a subset of $[\Delta+1]$, the output coloring will be a $(\Delta+1)$-coloring of the entire graph, concluding the proof. ☐

To obtain the desired algorithm in Theorem 1 (modulo the fact that so far we only focused on the queries and not runtime), we can just do the following: if $\Delta \leqslant \sqrt{n} \log n$, we run the standard greedy algorithm and query every entry of the adjacency list of the graph which requires $O(n\Delta) = O(n^{3/2} \log n)$ queries. Otherwise, when $\Delta > \sqrt{n} \log n$, we run Algorithm 1 which now requires $O(n^2 \log^2 n/\Delta) = O(n^{3/2} \log n)$ queries given the lower bound on Δ. This gives us an $O(n^{3/2} \log n)$ query algorithm for $\Delta+1$ coloring already! Of course, we are still quite far from being done, because we need to prove the Palette Sparsification Theorem itself. We will do that in the rest of this lecture, starting with some warm-ups.

3 Warm-up: Palette Sparsification in Special Cases

Sparse case. Let us consider a much simpler case of Theorem 1, wherein, *every* vertex, except for one, have degree $\leqslant \Delta/2$; in other words, while the maximum degree is Δ, (essentially) all vertices are "far from" having degree Δ. We refer to this as the "sparse" case of the theorem. The following lemma gives a quick proof of this case.

Lemma 6. *Suppose $G = (V, E)$ is a graph with maximum degree Δ, wherein all but a single vertex have degree at most $\Delta/2$. Suppose for every vertex $v \in V$, we sample a list $L(v)$ of colors of size $s = 8 \log n$ independently and uniformly at random from the colors $\{1, \ldots, \Delta+1\}$. Then, with high probability, G is list-colorable from lists $L(v)$ of $v \in V$.*

Proof. Order the vertices arbitrarily and then put the maximum degree vertex at the beginning of this ordering. Let $v_1, v_2, \ldots, v_n$ be this ordering of the vertices. We find a list-coloring of G as follows. Color v_1 using any color from its list and iterate over the vertices in this order and suppose we are at vertex v_i now. Given that degree of v_i is at most $\Delta/2$, at most $\Delta/2$ out of $\Delta+1$ colors have been used to color neighbors of v_i so far. As such, there are at least $\Delta/2$ *available* colors for coloring v_i: as long as $L(v_i)$ samples any of these colors, we will be able to color v_i also. Also, since the choice of colors are independent across different vertices, we can think of sampling the colors of v_i only now; this gives us

$$\Pr\left(L(v_i) \text{ contains no available color}\right) \leqslant \left(1 - \frac{\Delta}{2 \cdot (\Delta+1)}\right)^s \leqslant 2^{-8 \log n} = n^{-8}.$$

Now, a union bound over all n vertices implies that with high probability, every vertex of the graph will be colored correctly from its sampled list, concluding the proof. ☐

Dense case. Let us now consider another simple case of Theorem 1 that is somewhat the exact opposite of the sparse case we examined earlier. What if G is a $(\Delta + 1)$-clique?

It is easy to see that the proof of Lemma 6 no longer holds here: by the time we get to color the last vertex of the clique, all of its Δ neighbors should have necessarily been colored with distinct colors (or the algorithm may have failed already). But since we only have $(\Delta + 1)$ colors to begin with, this leaves out just a single good color for v; the probability that this color is chosen in $L(v)$ is then only $\Theta(\log n/\Delta)$ and thus with probability $1 - o(1)$, there is no color available to v.

Nevertheless, in the following lemma, we show that there is an entirely different argument that can handle this case also.

Lemma 7. *Suppose for every vertex v of a $(\Delta + 1)$-clique G, we sample a list $L(v)$ of colors of size $s = 8 \ln n$ independently and uniformly at random from the colors $\{1, \ldots, \Delta + 1\}$. Then, with high probability, G is list-colorable from lists $L(v)$ of $v \in V$ (note that here $n = \Delta + 1$).*

Proof. As we saw, there is no hope of list-coloring G greedily in this lemma. We instead use a "global" argument that allows us to color all vertices at the same time by exploiting their dependencies more carefully.

Define the following bipartite graph $\mathcal{G} := (\mathcal{L}, \mathcal{R}, \mathcal{E})$ which we call the **palette graph** (see Figure 2):

1. The set of vertices on the left $\mathcal{L}$ is all vertices of the $(\Delta + 1)$-clique;
2. The set of vertices on the right $\mathcal{R}$ is the colors $\{1, \ldots, \Delta + 1\}$;
3. There is an edge between a vertex $v \in \mathcal{L}$ to a color $c \in \mathcal{R}$ iff $c \in L(v)$.

We claim that there is a list-coloring of G from lists L iff $\mathcal{G}$ admits a perfect matching[4]: this is simply because there is a one-to-one mappings between list-colorings of G and perfect matchings in $\mathcal{G}$ by considering any edge (v, c) as part of the matchings of $\mathcal{G}$ as the assignment of color c from $L(v)$ to v in G.

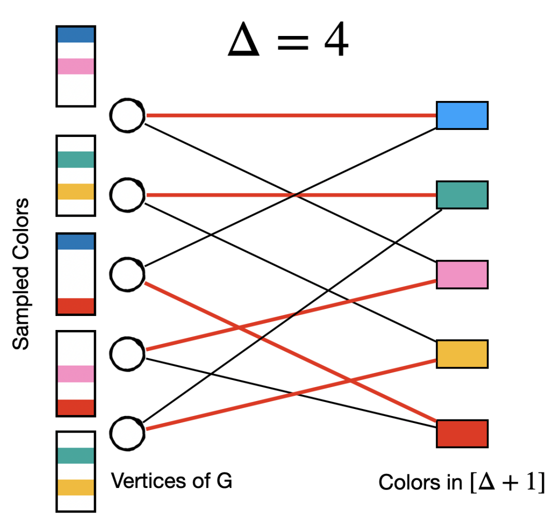


Figure 2: An illustration of the palette graph in Lemma 7: Finding a perfect matching in this palette graph allows us to color all vertices the same way in our clique.

It thus remains to prove that $\mathcal{G}$ admits a perfect matching with high probability. Considering $\mathcal{G}$ is a *random* graph with degree $\Theta(\log n)$, standard results in random graph theory already implies the existence of such a perfect matching [Bol01]. For completeness and as a warm up, we prove this claim below. We just need a quick reminder on Hall's theorem first.

[4]Recall that a matching is a collection of vertex-disjoint edges and a perfect matching is a matching that matches all vertices.

Proposition 8. *Let $G = (L, R, E)$ be any bipartite graph with $|L| = |R|$. There is a perfect matching in G if and only if for every set $S \subseteq L$, we have $|N(S)| \geqslant |S|$ in G.*

We will not prove this statement in this lecture even though its proof is quite simple and elementary (you can prove this using different ways: by induction directly, by using max-flow min-cut duality, or by using matching-vertex cover duality we covered in Lecture 2). We use this to prove that the palette graph has a perfect matching almost surely.

Claim 9. *The bipartite graph $\mathcal{G} := (\mathcal{L}, \mathcal{R}, \mathcal{E})$ has a perfect matching with high probability.*

Proof. By Hall's theorem (Proposition 8) for this bipartite graph to not have a perfect matching, there should exist a set S in $\mathcal{L}$, and T in $\mathcal{R}$, such that $|T| = |S| - 1$ and there are no edges between S and $\bar{T} = \mathcal{R} \setminus T$ (this is true iff $|N(S)| < |S|$ for some S which is the more familiar statement of Hall's theorem).

Let us now fix such a choice of $S \subseteq \mathcal{L}$ and $T \subseteq \mathcal{R}$ and see what is the probability of this event happening:

$$\Pr\left(N_{\mathcal{G}}(S) \subseteq T\right) \leqslant \left(\frac{|T|}{n}\right)^{|S| \cdot s} = \left(\frac{|S|-1}{n}\right)^{|S| \cdot s} = \left(1 - \frac{n - |S| + 1}{n}\right)^{|S| \cdot s} \leqslant \exp\left(-\frac{(n - |S| + 1) \cdot |S|}{n} \cdot s\right).$$

We can bound the above quantity differently for $|S| \leqslant n/2$ and $|S| > n/2$ and obtain that:

$$\text{when } |S| \leqslant n/2\text{:} \quad \Pr\left(N_{\mathcal{G}}(S) \subseteq T\right) \leqslant \exp\left(-|S| \cdot 4 \ln n\right);$$
$$\text{when } |S| > n/2\text{:} \quad \Pr\left(N_{\mathcal{G}}(S) \subseteq T\right) \leqslant \exp\left(-(n - |S| + 1) \cdot 4 \ln n\right);$$

Finally, we do a union bound over all choices of S, T, by grouping them based on the size of S:

$$\begin{aligned}
\Pr\left(\mathcal{G} \text{ has no perfect matching}\right) &\leqslant \sum_{k=1}^{n} \sum_{\substack{S \subseteq \mathcal{L} \\ |S| = k}} \sum_{\substack{T \subseteq \mathcal{R} \\ |T| = k-1}} \Pr\left(N_{\mathcal{G}}(S) \subseteq T\right) \\
&\leqslant \sum_{k=1}^{n/2} \binom{n}{k}^2 \exp\left(-4k \cdot \ln n\right) + \sum_{k=n/2+1}^{n} \binom{n}{k-1}^2 \exp\left(-4(n-k+1) \cdot \ln n\right) \\
&\qquad \text{(by monotonicity of } \tbinom{a}{b} \text{ for } b \leqslant a/2 \text{ and } b \geqslant a/2 \text{ individually)} \\
&= \sum_{k=1}^{n/2} \binom{n}{k}^2 \exp\left(-4k \cdot \ln n\right) + \sum_{k=n/2+1}^{n} \binom{n}{n-k+1}^2 \exp\left(-4(n-k+1) \cdot \ln n\right) \\
&\qquad \text{(as } \tbinom{a}{b} = \tbinom{a}{a-b}\text{)} \\
&\leqslant \sum_{k=1}^{n/2} \exp\left(-2k \cdot \ln n\right) + \sum_{k=n/2+1}^{n} \exp\left(-2(n-k+1) \cdot \ln n\right) \qquad \text{(as } \tbinom{a}{b} \leqslant a^b\text{)} \\
&= 2 \sum_{k=1}^{n/2} \frac{1}{n^{2k}} \leqslant \frac{3}{n^2}. \qquad \square_{\text{ Claim 9}}
\end{aligned}$$

We are now done as we can color the vertices of G according to the perfect matching of $\mathcal{G}$ and find a list-coloring of G from the lists $\{L(v) \mid v \in G\}$. $\square$

The natural question at this point is that whether we can extend the quite different strategies we saw for proving Theorem 2 in sparse and dense regimes, to prove the whole theorem as well. We will do so in the next section.

4 Proof of the Palette Sparsification Theorem

We now present a proof of the palette sparsification theorem. The general strategy of the proof is as follows: the greedy list-coloring approach to coloring G from the sampled lists $\{L(v) \mid v \in V\}$ works fine for coloring vertices of degree $(1-\Omega(1)) \cdot \Delta$ using the same proof as Lemma 6; as we shall see, this proof can be extended even to subgraphs of G that are "locally sparse", i.e., with "sufficient" number of non-edges in the subgraph. At the same time, to color the subgraphs of G that are "almost a clique", we can use a similar strategy as the random graph theory approach of Lemma 7. Thus for a complete proof, we are going to decompose the graph into sparse and dense subgraphs and apply each of these ideas separately to each part, while showing that the resulting partial colorings of each part can be extended to the entire graph.

Remark. This approach of coloring sparse and dense parts of the graph separately has a rich history in the graph theory literature starting from the pioneering work of Reed [Ree98]. The very high level idea is typically as follows: the sparse vertices do not have much structure but can be colored greedily while the dense part typically cannot be colored greedily but has a "nice" structure that one can exploit in a more global argument[a].

[a]The reason this global argument is not applied to the entire graph is the lack of any structure for sparse vertices which makes the arguments tedious or downright infeasible to carry out there.

4.1 A Sparse-Dense Decomposition

We shall use the following decomposition which is a simple extension of the classical graph decomposition of Reed in [Ree98] and numerous follow-ups (see Chapter 15 of the excellent book of Molloy and Reed on graph coloring [MR02] for more details).

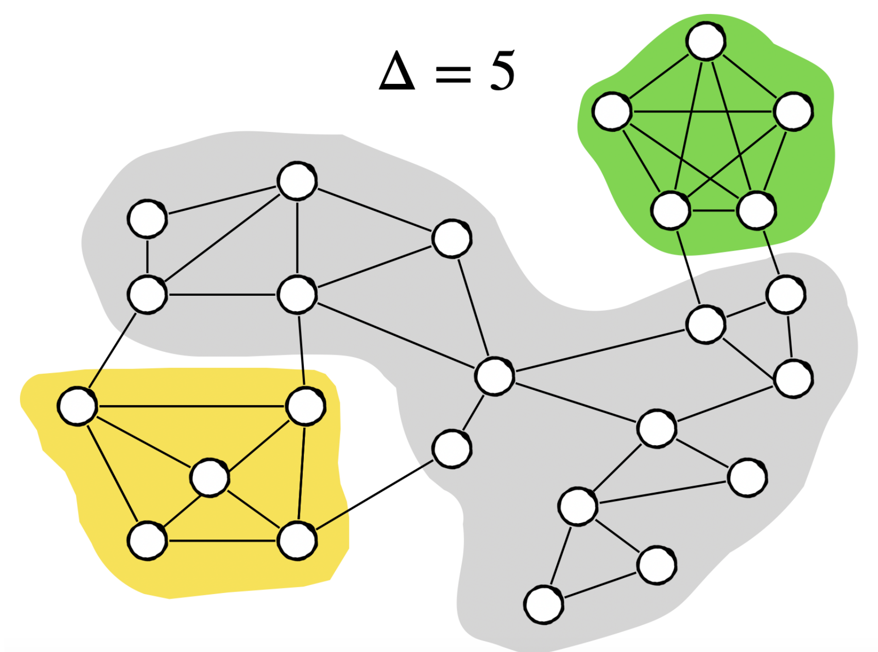


Figure 3: An illustration of the decomposition in Proposition 10. The vertices in the middle (gray) are all sparse vertices while the other two components (green and yellow), each form an almost-clique. Roughly speaking, a sparse vertex is such that its neighborhood is "far from" being a $(\Delta+1)$-clique, while an almost-clique is a subgraph which is "close to" being a $(\Delta+1)$-clique.

Proposition 10. *Let $\varepsilon < 1/50$ and $G = (V, E)$ be any arbitrary graph. We can decompose the vertices of G into the sets $V_{sparse}, C_1, \ldots, C_k$ with the following properties:*

(i) *For every vertex $v \in V_{sparse}$, the total number of edges between the neighbors of v is $(1-\varepsilon^2)\cdot\Delta^2/2$;*

(ii) *Any set of vertices C_i, called an* **almost-clique**[5]*, has the following properties:*

- *(a)* $(1-5\varepsilon)\Delta \leqslant |C_i| \leqslant (1+5\varepsilon)\Delta$;
- *(b) Any vertex $v \in C_i$ has at most $10\varepsilon\Delta$ neighbors outside of C_i;*
- *(c) Any vertex $v \in C_i$ has at most $10\varepsilon\Delta$ non-neighbors inside of C_i.*

Proof. Let D denote the set of all vertices with more than $(1-\varepsilon^2)\cdot\Delta^2/2$ edges among their neighbors. For any vertex $v \in D$, define the following set S_v:

1. Let $S_v \leftarrow N(v) \cup \{v\}$ initially;
2. While there is a vertex $u \in S_v$ with $|N(u) \cap S_v| \leqslant (1-4\varepsilon)\cdot\Delta$, delete u from S_v;
3. While there is a vertex $w \in V \setminus S_v$ with $|N(w) \cap S_v| \geqslant (1-4\varepsilon)\cdot\Delta$, insert w to S_v.

Considering there is no alternation between deleting and inserting vertices to S_v, this process gives a unique set S_v. We now list the following properties of the sets S_v for all $v \in D$:

(i) every vertex $u \in S_v$ has at least $(1-4\varepsilon)\Delta$ neighbors in S_v;

Proof: by the definition of the set S_v itself.

(ii) every vertex $u \in S_v$ has at most $4\varepsilon\Delta$ neighbors outside of S_v;

Proof: by part (i) and the fact that degree of each vertex is at most Δ.

(iii) $|N(v) \cap S_v| \geqslant (1-3\varepsilon)\Delta$ and at least $(1-3\varepsilon)\Delta$ vertices in $N(v)$ have $\leqslant 3\varepsilon\Delta$ neighbors outside of S_v.

Proof: we define the non-degree of $u \in N(v)$ as the number of vertices in $N(v)$ that are not neighbor to u. As the average degree of vertices in $N(v)$ is $\geqslant (1-\varepsilon^2)\Delta$ and $|N(v)| \leqslant \Delta$, the average non-degree is $\leqslant \varepsilon^2\Delta$. By Markov bound, number of vertices $U \subseteq N(v)$ with non-degree $> \varepsilon\Delta$ is less than $\varepsilon\Delta$. As such, each vertex of $N(v) \setminus U$ has more than $|N(v)| - \varepsilon\Delta - \varepsilon\Delta > (1-3\varepsilon)\Delta$ neighbors inside $N(v) \setminus U$ (note that $|N(v)| > (1-2\varepsilon^2)\Delta$ as otherwise v cannot belong to D even if its neighborhood is a clique). Finally, it can be seen that $S_v \supseteq N(v) \setminus U$ as none of the vertices of $N(v) \setminus U$ would ever be deleted in the definition of S_v, implying the property.

(iv) $|S_v \setminus N(v)| \leqslant 5\varepsilon\Delta$ and the total number of edges going out of S_v is $\leqslant 4\varepsilon\Delta^2$.

Proof: consider the set S_v at the end of the step (2) of its definition. At this point, the total number of edges going from S_v to outside vertices by parts (iii) and (ii) is $\leqslant (1-3\varepsilon)\Delta\cdot 3\varepsilon\Delta + 3\varepsilon\Delta\cdot 4\varepsilon\Delta < 4\varepsilon\Delta^2$. At the same time, whenever we add a new vertex w to S_v in the step (3), we reduce the number of outgoing edges by $\geqslant (1-4\varepsilon)\Delta$ due to the inclusion of w, and increase it by $\leqslant 4\varepsilon\Delta$ for the new contribution of the outside edges of w. As such, the total number of times we can include a new vertex w to S_v at step (3) is $(4\varepsilon\Delta^2)/(1-8\varepsilon)\Delta \leqslant 5\varepsilon\Delta$. This means that $|S_v \setminus N(v)| \leqslant 5\varepsilon\Delta$.

(v) If $S_v \cap S_u \neq \emptyset$, then u belongs to S_v and v belongs to S_u;

Proof: let w be any vertex in $S_v \cap S_u$ and note that by part (i), $N(w)$ intersects both S_v and S_u in at least $(1-4\varepsilon)\Delta$ vertices. By part (iv), this also implies that

$$|N(v) \cap N(w)| \geqslant (1-4\varepsilon-5\varepsilon)\Delta > 4\Delta/5 \quad \text{and} \quad |N(u) \cap N(w)| > (1-4\varepsilon-5\varepsilon)\Delta > 4\Delta/5.$$

[5]The reason behind the name should be clear from the properties; each C_i can be turned into a $(\Delta+1)$-clique by changing $\approx \varepsilon$ fraction of its vertices and edges.

As such $|N(u) \cap N(v)| \geqslant 3\Delta/5$. Now, consider the set of edges between $N(v) \cap N(u)$ and $N(u) \setminus N(v)$. On one hand, all these edges are from $N(v)$ to outside $N(v)$ and there can be only $\varepsilon^2\Delta^2$ such edges since $v \in D$. On the other hand, all these edges are inside $N(u)$ and there are at most $\varepsilon^2\Delta^2$ non-edges inside $N(u)$ since $u \in D$. This implies that

$$|N(v) \cap N(u)| \cdot |N(u) \setminus N(v)| - \varepsilon^2\Delta^2 \leqslant |E(N(v) \cap N(u), N(u) \setminus N(v))| \leqslant \varepsilon^2\Delta^2.$$

Combining with the (loose) lower bound of $|N(u) \cap N(v)| \geqslant 3\Delta/5$, we get $|N(u) \setminus N(v)| < 4\varepsilon^2\Delta$. This gives a much stronger lower bound of $|N(v) \cap N(u)| > (1-\varepsilon)\Delta$. Finally, this means that

$$|N(v) \cap S_u| > (1-4\varepsilon)\Delta \quad \text{and} \quad |N(u) \cap S_v| > (1-4\varepsilon)\Delta,$$

implying that $v \in S_u$ and $u \in S_v$ by steps (2)/(3) of the procedure.

(*vi*) v belongs to S_v;

Proof: By part (*iii*), $|N(v) \cap S_v| > (1-3\varepsilon)\Delta$ and thus v never gets deleted from S_v in step (2).

Let us now construct the decomposition. We go over vertices of D in some arbitrary order and pick $v \in D$ and corresponding S_v to form an almost-clique C; we then remove all of S_v from D. Properties (*v*) and (*vi*) ensures that the collection of S_v-sets picked in this procedure partitions D and that each almost-clique is corresponding to some set S_v. This ensures that for any set C ($= S_v$ for some $v \in D$):

- $(1-3\varepsilon)\Delta \leqslant |C| \leqslant (1+5\varepsilon)\Delta$: by properties (*iii*) and (*iv*) of $S_v = C$;
- each vertex $u \in C$ has at most $4\varepsilon\Delta$ outside neighbors: by property (*ii*) of $S_v = C$;
- each vertex $u \in C$ has at most $9\varepsilon\Delta$ inside non-neighbors: by property (*i*) and (*iv*) of $S_v = C$.

As a result each C is truly an almost-clique as desired in Proposition 10. Finally, any remaining vertex after this step does not belong to D and thus can be safely placed in V_{sparse}, concluding the proof. □

Throughout the proof of Theorem 2, we pick $\varepsilon := 10^{-10}$, chosen so that any constant arising in the proofs times $\varepsilon < 1$ and we do not mention this explicitly each time. We then fix a decomposition of the input graph $G = (V, E)$ with this parameter ε given by Proposition 10 and let V_{sparse} and $C_1, \ldots, C_k$ refer to the sets of this decomposition. We assume the number of sampled colors is $O(\varepsilon^{-4} \cdot \log n)$ (which is $\Theta(\log n)$ considering $\varepsilon = \Theta(1)$). We also assume that $\Delta = \Omega(\varepsilon^{-4} \cdot \log n)$ as otherwise sampling the colors in Theorem 2 will collect all available colors of each vertex and the theorem become vacuously true[6].

Remark. The decomposition given in Proposition 10 is somewhat different, in terms of the constructions but not the properties, than the one used in the original proof of [ACK19], which was motivated by the HSS network decomposition of [HSS16]. The reason for the choice of [ACK19] is that it is easier to obtain the decomposition using the existing result of [HSS16] compared to the approach above, and the resulting decomposition is more suitable for the "algorithmic" palette sparsification that is needed to obtain fast algorithms for the list-coloring part.

[6]We note that as stated, the hidden constant in $O(\log n)$ samples for this theorem is $\sim 10^{40}$ or so (!). However, we made no attempt to reduce the constants and simply used the simplest choices throughout to help clarify the (already complicated enough) proof. Indeed, one can reduce these constants dramatically without changing any part of the proof; that being said, the limit of this approach is still a large constant $\sim 10^3$ or so.

Very recently, Kahn and Kenney [KK23] generalized and modified this argument (still following the sparse-dense decomposition approach) to obtain the *right* constant for this problem: sampling $(1 + o(1)) \cdot \ln(n)$ colors is necessary and sufficient for palette sparsification!

4.2 Part One: Coloring Vertices in V_{sparse}

We first show that all vertices in V_{sparse} are list-colorable from the sampled lists with high probability. The proof of this part closely follows the classical approaches in graph theory [MR97] for proving that "locally sparse" vertices, as in V_{sparse}, can be colored with much less than Δ colors. The approach is to first show that we can color a constant fraction of vertices $v \in V_{sparse}$ using colors of $L(v)$ in a way that the remaining uncolored graph basically becomes a "low degree" graph and then apply the same argument as in Lemma 6 to finalize the proof. We note that throughout this proof, we solely focus on the induced subgraph of G on V_{sparse} (notice that property of V_{sparse} vertices in Proposition 10 continues to hold here as well as we may have only reduced the number of edges in their neighborhoods).

Part One-(A): Creating Excess Colors

Consider the following algorithm for a partial list-coloring of V_{sparse}:

Algorithm 2. An algorithm for partially list-coloring a subset of V_{sparse} from the sampled lists.

(*i*) For any vertex $v \in V_{sparse}$, let $c(v)$ be the first colored sampled in $L(v)$.

(*ii*) If the color $c(v) \neq c(u)$ for any neighbor $u \in N(v)$, we **color** v with $c(v)$.

For any vertex $v \in V_{sparse}$, define:

- $A_1(v)$: the colors in $\{1, \ldots, \Delta+1\} - c(v)$ that are not used to color any neighbor of v in Algorithm 2; these colors are *available* to v (if included in $L(v)$), for the next step of coloring V_{sparse}.
- $\deg_1(v)$: the number of uncolored neighbors of v after Algorithm 2.

The intuition behind Algorithm 2 is simply as follows: for any pairs of vertices in the neighborhood of a vertex $v \in V_{sparse}$ that are **colored the same**, $A_1(v)$ reduces by one while $\deg_1(v)$ reduces by two. As such, if we can have a "large" number of pairs of vertices in the neighborhood of v with the same color, then v becomes "low degree" compared to the number of available colors it has (and then we know how to handle low degree vertices similar to Lemma 6 in the next part). Now considering the fact that there are many non-adjacent pairs of vertices in $N(v)$ as $v \in V_{sparse}$, we hope that Algorithm 2 would be able to color enough of number of such pairs with the same color[7]. We now formalize this as follows.

Lemma 11. *After running Algorithm 2, with high probability,* $|A_1(v)| \geqslant \deg_1(v) + 10^{-4} \cdot \varepsilon^2\Delta$.

Proof. For the purpose of this proof, let us assume that degree of v is exactly Δ: this can be achieved by adding a set of $\Delta - \deg(v)$ 'dummy' vertices to the neighborhood of v. Considering v belongs to V_{sparse}, by Proposition 10, the number of edges in the neighborhood of v is at most $(1-\varepsilon^2) \cdot \Delta^2/2$, which implies that there are $\geqslant \binom{\Delta}{2} - (1-\varepsilon^2)\Delta^2/2 = \varepsilon^2\Delta^2/2 - \Delta/2$ **non-edges** between neighbors of v (including the new dummy vertices). Let $t = \varepsilon^2\Delta^2/3$ and $\bar{f}_1, \ldots, \bar{f}_t$ denote t arbitrary non-edges in $N(v)$. Additionally, define the following random variable:

- X: number of colors that are sampled by *at least* two neighbors of v and are additionally retained by *all* these neighbors.

Since any color counted in X is used more than once to color a neighbor of v, we have

$$|A_1(v)| \geqslant \deg_1(v) + X - 1, \tag{1}$$

[7] Strictly speaking, the statement that "there are many non-adjacent pairs of vertices in $N(v)$ as $v \in V_{sparse}$" is not true as when $\deg(v) < (1-\varepsilon)\Delta$, $N(v)$ can be a clique and v would still be in V_{sparse}. But in that case v is a already low degree and there is nothing we need to do anyway.

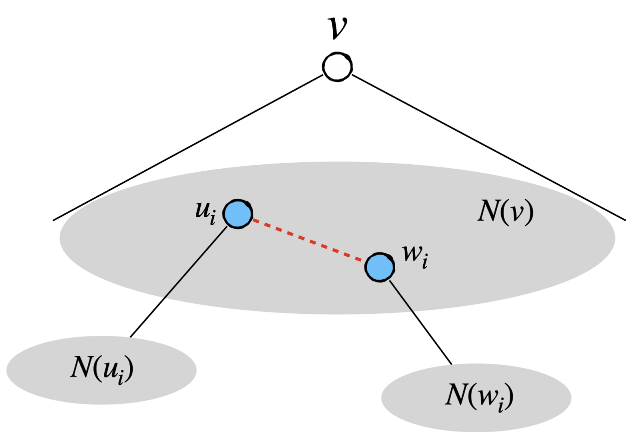


Figure 4: An illustration of the intuition behind Algorithm 2 and the proof of Lemma 11: the more pairs of vertices we can color the same in $N(v)$, the more colors we "save" for the next step. A fixed pairs of vertices (u_i, w_i) in $N(v)$ with no edges between them will be colored the same as long as $c(u_i) = c(w_i)$ and this color is not sampled in any of $N(v)$, $N(u_i)$ or $N(w_i)$ – this happens with probability $\Theta(1/\Delta)$.

where the -1 term is because of $c(v)$ itself. Let us first lower bound the expected value of X.

Claim 12. $\mathbb{E}[X] \geqslant e^{-6} \cdot \varepsilon^2 \cdot \Delta$.

Proof. We define Y as the number of colors that are sampled by the endpoints of exactly one of $\bar{f}_i$'s and are retained by both endpoints. Clearly, $X \geqslant Y$. We thus lower bound Y instead. For every non-edge $\bar{f}_i := (u_i, w_i)$, define the indicator random variable $\bar{f}_i$ where $\bar{f}_i = 1$ if the following two conditions happen:

- $c(u_i) = c(w_i)$, and,
- for all $z \in N(v) \cup N(u_i) \cup N(w_i) \setminus \{u_i, w_i\}$, $c(z) \neq c(u_i)$ (this way, $c(u_i) = c(w_i)$ and is assigned to both of them for sure);

otherwise $\bar{f}_i = 0$. By definition, $Y = \sum_{i=1}^{t} Y_i$. We have,

$$\Pr[Y_i = 1] = \Pr[c(u_i) = c(w_i) \wedge \forall z \in N(u_i) \cup N(w_i) \cup N(v) \setminus \{u_i, v_i\} : c(z) \neq c(u_i)]$$

$$\geqslant \frac{1}{\Delta+1} \cdot \left(1 - \frac{1}{\Delta+1}\right)^{\Delta+(\Delta-1)+(\Delta-2)}$$

(the color of each vertex is chosen independently and uniformly at random from $\Delta + 1$ colors)

$$\geqslant \frac{1}{\Delta+1} \cdot \exp\left(-\frac{4\Delta}{\Delta+1}\right) \geqslant \frac{e^{-4}}{\Delta+1}.$$

($1 - x \geqslant e^{-4x/3}$ for sufficiently small $x \in (0, 1/10)$ and since Δ is $\omega(1)$)

By linearity of expectation, $\mathbb{E}[X] \geqslant \mathbb{E}[Y] \geqslant t \cdot \frac{e^{-4}}{\Delta+1} \geqslant e^{-6} \cdot \varepsilon^2 \Delta$ as $t = \varepsilon^2 \Delta^2/3$. $\square$ Claim 12

The next step is to prove that X is concentrated. The proof of this part is a rather involved yet standard exercise in probabilistic arguments (see, e.g. [MR02, Chapter 10]) and can be skipped by the reader.

The proof uses Talagrand's inequality: A function $f(x_1, \ldots, x_n)$ is called ***c*-Lipschitz** if changing any x_i can affect the value of f by at most c. Additionally, f is called ***r*-certifiable** iff whenever $f(x_1, \ldots, x_n) \geqslant s$, there exists at most $r \cdot s$ variables $x_{i_1}, \ldots, x_{i_{r \cdot s}}$ so that knowing the values of these variables certifies $f \geqslant s$.

Proposition 13 (Talagrand's inequality; cf. [MR02]). *Let $X_1, \ldots, X_n$ be n independent random variables and $f(X_1, \ldots, X_n)$ be a c-Lipschitz and r-certifiable function. For any $t \geqslant 1$,*

$$\Pr\left(|f - \mathbb{E}[f]| > t + 30c\sqrt{r \cdot \mathbb{E}[f]}\right) \leqslant 4\exp\left(-\frac{t^2}{8c^2 r\,\mathbb{E}[f]}\right).$$

Remark. Perhaps the most common concentration inequality used in probabilistic analysis is Chernoff bound. Despite its simplicity, Chernoff bound is applicable to a surprisingly large number of situation. Yet, there are still some limits to the power of this basic tool. In such cases, one may want to consider stronger tools such as Talagrand's, McDiarmid's, or Azuma's inequalities. At a very high level, the common theme of all these concentration inequalities is the following: if we have a random variable T which is a function of n independent trials $X_1, \ldots, X_n$ and T is not "very sensitive" to the outcome of each trial[a], then T is "more or less" concentrated around its expectation.

[a]This is captured by the notions of Lipschitz constant and certifiably in Talagrand's inequality in Proposition 13.

We still cannot apply Proposition 13 directly to bound the deviation of X as X is not a certifiable function of the independent random variables in this context, i.e., $c(v)$ for $v \in V_{sparse}$. As such, we prove the concentration of X indirectly as follows. Define the following two additional variables:

- W: number of colors that are sampled by *at least* two neighbors of v, regardless of whether they are assigned to them or not.
- Z: number of colors that are sampled by *at least* two neighbors of v but are *not* assigned to *at least one* of them.

It is clear that $X = W - Z$. Also notice that both W and Z are functions of *independent* random variables that define the choices of random colors $c(v)$ for every $v \in V_{sparse}$. Both W and Z are $\Theta(1)$-certifiable (for W point to two neighbors of v that sampled the color; for Z additionally point to one of the neighbors of this pair that also sampled the color, hence not allowing one of them to retain it). They are also both $\Theta(1)$-Lipschitz: changing choice of one color for a vertex can only affect the two colors involved (the original one and the changed one). As such, we can apply Talagrand's inequality (Proposition 13) to obtain that (we only write the bound for W; the same exact argument works also for Z):

$$\Pr\left(|W - \mathbb{E}[W]| \geqslant \mathbb{E}[X]/10\right) \leqslant \exp\left(-\Theta(1) \cdot \frac{(\mathbb{E}[X] - \Theta(1)\sqrt{\Delta})^2}{\Delta}\right) \qquad (W \leqslant \Delta/2 \text{ always})$$
$$\leqslant \exp\left(-\Theta(\varepsilon^4) \cdot \Delta\right) \ll n^{-10}.$$
(by Claim 12 on expected value of X and since we can assume Δ to be $\gg \varepsilon^{-4} \cdot \ln n$)

As such, we obtain that w.h.p. both W and Z are concentrated and thus also w.h.p.,

$$X = W - Z \geqslant \mathbb{E}[W] - \mathbb{E}[X]/10 - (Z + \mathbb{E}[X]/10) = \mathbb{E}[X] - \mathbb{E}[X]/5 = (4/5) \cdot \mathbb{E}[X].$$

Plugging in this and Claim 12 in Eq (1) implies that $|A_1(v)| \geqslant \deg(v) + e^{-7} \cdot \varepsilon^2 \Delta$ with high probability. A union bound over all vertices finalizes the proof as $e^7 < 10^4$. □

Part One-(B): Finalizing the Coloring of V_{sparse}

In the following, we condition on the high probability event of Lemma 11. We can now use the same greedy algorithm in the proof of Lemma 6 to finalize the coloring of vertices in V_{sparse}.

Algorithm 3. An algorithm for finalizing the list-coloring of V_{sparse} from the sampled lists.

(i) Go over uncolored vertices v in V_{sparse} in some arbitrary order;

(ii) For any vertex v, if $L(v) = \emptyset$ output FAIL; otherwise, **color** v from any color c in $L(v)$ and remove the color c from $L(u)$ for any $u \in N(v)$.

Lemma 14. *With high probability, Algorithm 3 does not output FAIL.*

Proof. Suppose we are at the step of coloring a vertex $v \in V_{sparse}$. Recall that at the beginning of Algorithm 3, the list of available colors to v were $A_1(v)$ and by the time we get to process v, at most $\deg_1(v)$ other colors have been removed from $A_1(v)$, leaving us with at least $10^{-4} \cdot \varepsilon^2\Delta$ choices by Lemma 11.

Now, if $L(v)$ contains any of these colors, we will be able to color v and no need to output FAIL at this point. By the independence of the choice of lists, we can think of $L(v) - c(v)$ (where $c(v)$ is the color used in Algorithm 2) as being sampled only now. As such, by considering the $s-1$ random colors remained in $L(v)$ at this point, we have,

$$\Pr\left(L(v) \text{ has no color for coloring } v\right) \leqslant \left(1 - \frac{10^{-4} \cdot \varepsilon^2\Delta}{\Delta+1}\right)^{(s-1)} \leqslant \exp\left(-10^{-4} \cdot \varepsilon^{-2}(s-1)/2\right) \leqslant n^{-5}.$$

(by sampling $s \geqslant 10^{-5} \cdot \varepsilon^2 \log n$ colors in $L(v)$ which is $O(\log n)$ still)

A union bound over all $v \in V_{sparse}$ implies that the algorithm does not output FAIL with high probability. $\square$

Lemmas 11 and 14 now imply that we can list-color all vertices in V_{sparse} from the lists $L(v)$ with high probability, finishing the first part of the proof of Theorem 2.

4.3 Part Two: Coloring Almost-Cliques One by One

We now start the main part of the proof of Theorem 2 which concerns coloring each almost-clique. Our approach would be to go over each almost-clique C one by one, assume that every vertex outside C is colored (even adversarially), and show that the randomness in the choice of $L(v)$ for $v \in C$ is still enough to list-color C in a way that it is also consistent with the coloring of outside vertices. The main idea is to use a generalization of the random graph theory approach used in Lemma 7. However, this can be tricky when working with an almost-clique that contains "many" non-edges inside[8]. As a result, the first step of our approach here is a pre-processing step for almost-cliques with a "'large" number of non-edges.

Throughout this section, we fix an almost-clique C in the decomposition of Proposition 10; fix the choice of all lists $L(v)$ for $v \notin C$, and assume that the subgraph $G - C$ is already colored, even adversarially. We then show that with high probability, we can still color C from the sampled lists $L(v)$ for $v \in C$ in a way that it is consistent with the coloring of $G - C$. Having achieved this, we can go over almost-cliques of G one by one and use this approach to color them.

A note on sampling the lists. We are going to *partition* the sampled lists into $k := \Theta(\log n)$ lists $L_1(v), \ldots, L_k(v)$ plus another list $L^*(v)$ such that (1) each color belongs to $L_i(v)$ for $i \in [k]$ with probability $q := 1/(100 \cdot \sqrt{\varepsilon} \cdot \Delta)$ (this requires $|L_i(v)| = \Theta(1/\sqrt{\varepsilon})$ but we do not calculate the exact constant here[9]; and (2) have $L^*(v) = \Theta(\log n)$ (we will specify the constant of Θ-notation in Lemma 21). Thus, we can think of

$$L(v) := (L_1(v), \ldots, L_k(v), L^*(v)).$$

Given that the colors in $L(v)$ are sampled with repetition, this is without loss of generality.

[8]Among other challenges, such an almost-clique may even have more than $\Delta+1$ vertices. In Lemma 7, we modeled the problem as a matching problem—when number of vertices is $> \Delta+1$ we certainly cannot hope to get a matching as some colors need to be assigned to at least two vertices; the vertices that are colored the same should also be an independent set in the original graph, a constraint that makes using the random graph theory approach rather infeasible.

[9]One can calculate the constant ℓ through the equation $q = 1 - (1 - 1/(\Delta+1))^{\ell}$ which is roughly $\frac{\ell}{\Delta}$.

Part Two-(A): Pre-Processing the Almost-Clique C

We follow a similar high level strategy as Part One-(A), by trying to color *two* vertices in the almost-clique C with the same color, hence "buying" ourselves some extra colors for the main part of the argument. However, the way this coloring is obtained and its properties are entirely different from that of Algorithm 2. In particular, this time we need to able to exploit the case when we have only $\approx \varepsilon\Delta$ non-edge in an almost-clique compared to the case of Part One-(A) that worked with $\approx \varepsilon^2\Delta^2$ non-edges. On the other hand, the structure of almost-cliques allows us to do this more efficiently than the random coloring of Algorithm 2.

The general idea of the algorithm is to find a *matching* among the *non-edges* of C such that both endpoints of this non-edge can be colored the same (from a color shared in the list of both endpoints which also does not appear in the neighborhood of either outside C), while using a different color for each of its edges, namely, a **colorful non-edge matching**. Such an approach would reduces the number of vertices of C at twice the rate of the number of used colors, hence preparing C for the last step of our algorithm[10].

Let us briefly go over the intuition behind the algorithm first. Let t denote the number of non-edges in C and suppose $t \gtrsim \Delta$ (this is the regime we need the preprocessing for). For us to be able to include a non-edge $\bar{f} = (u, v)$ in $\bar{M}$ we need $L(u) \cap L(v) \neq \emptyset$—moreover, this intersection should be on a color c not used in the neighborhood of each vertex; considering out-degrees of u, v is $\approx \varepsilon\Delta$ by the decomposition of Proposition 10, we still have that $L(u) \cap L(v)$ is non-empty with probability $\approx 1/\Delta$ (as we calculated in Lemma 6; here, we are omitting the $O(\log n)$ factor as it is not relevant). Thus, we expect $\approx t/\Delta$ non-edges to be readily available to be added to $\bar{M}$. Moreover, considering the non-edge degree of vertices in C is $\approx \varepsilon\Delta$ by the decomposition of Proposition 10, after considering the non-edges with intersecting lists, we expect that there is a non-edge *matching* of size $\approx t/\Delta$. Finally, we need this matching to be colorful as well but one expects that the randomness in the choices of lists and their intersection over the non-edges of the matching already takes care of that, leading us to a non-edge matching of size $\approx t/\Delta$, which is sufficient for us.

Turning this intuition into an actual proof is tricky however considering the non-trivial correlations in the events above, and in particular as (i) the randomness is over the vertices of the graph (or rather their lists), while we would like to work with the non-edges, (ii) including some non-edge into the matching prohibits inclusion of several others due to the matching constraint, and (iii) the set of colors that can be used across non-edges is correlated due to the coloring of vertices outside the almost-clique. Consequently, we will turn the above intuition to an actual proof in a rather indirect way (this is also the reason we partitioned the lists of each vertex into $\Theta(\log n)$ parts and focused on each one individually for this step). In the following, we define our algorithm for each of the lists $L_i(v)$ for $i \in [k]$ and show that each list can "succeed" with constant probability; considering we have $\Theta(\log n)$ lists, at least one of them succeeds with high probability.

Algorithm 4. An algorithm for finding a "colorful non-edge matching" in the almost-clique C using only the lists $\{L_i(v) \mid v \in V\}$.

1. Let $\bar{F} := \bar{f}_1, \ldots, \bar{f}_t$ be the set of non-edges in C, i.e., pairs (u, v) in C with no edge between u , v.
2. Iterate over colors $c \in \{1, \ldots, \Delta + 1\}$ in an arbitrary order:
   (a) If there is $\bar{f} := (u, v) \in \bar{F}$ s.t. $c \in L_i(u) \cap L_i(v)$ and c is not assigned to any neighbor of u, v yet: add $\bar{f}$ to $\bar{M}$, remove all non-edges incident on u, v from $\bar{F}$, and continue to next color.

Let $\bar{M}$ denote the non-edge matching returned by Algorithm 4. We can use this non-edge matching to color endpoints of any $\bar{f} \in \bar{M}$ with the color c assigned to $\bar{f}$ when we added it to $\bar{M}$; as each color is given only for one non-edge and these colors both belong to the lists of the endpoints and at the same time have

[10] Again, for intuition, consider the case when $|C| = \Delta + 2$ and thus originally we cannot hope to get a matching of all vertices in C to the $\Delta + 1$ colors. However, such an almost-clique C necessarily has $\geqslant \Delta$ non-edges and so Part Two-(B) will be able to find a non-edge matching of size at least 1 inside C; by coloring the endpoints of this non-edge we will have $\Delta + 2 - 2 = \Delta$ remaining vertices to be colored and $\Delta + 1 - 1 = \Delta$ available colors; thus, at least finding a matching of remaining vertices to the colors is not completely out of the question.

not been used in the neighborhood of either, this would be a consistent coloring with the rest of the graph. As such, after this step, if we define $\mathsf{Remain}(C)$ as the set of remaining uncolored vertices in C and $\mathsf{Colors}(C)$ as the set of colors not used inside C still, we will have:

$$|\mathsf{Remain}(C)| = |C| - 2 \cdot |\bar{M}| \qquad , \qquad |\mathsf{Colors}(C)| = \Delta + 1 - |\bar{M}|. \tag{2}$$

We will only need Algorithm 4 for almost-cliques with a "non-trivial number" of non-edges which is $\gtrsim \varepsilon\Delta$. In that case, the following lemma shows that we can find a colorful non-edge matching which is smaller than the number of non-edges by roughly $\approx \varepsilon\Delta$.

Lemma 15. *Suppose C has at least $t \geqslant 10^7 \cdot \varepsilon\Delta$ non-edges in $\bar{F}$. Then, Algorithm 4 finds a colorful non-edge matching of size at least $\ell := 10^{-6} \cdot t/\varepsilon\Delta$ with probability at least $1/2$.*

Proof. In the following, we assume that the algorithm right away terminates if it finds a non-edge matching of size ℓ. For any color c when considered in Line (ii) of Algorithm 4, we define:

- $\mathsf{Present}(c)$: the set of non-edges in $\bar{F}$ at the time of processing c such that c is not used in the neighborhood of either endpoints, i.e.,

  $$\mathsf{Present}(c) := \left\{(u, v) \in \bar{F} \mid \text{no neighbors of } u \text{ or } v \text{ is colored with } c\right\};$$

  let $\mathsf{pre}(c) := |\mathsf{Present}(c)|$.

- We say that a color c is **successful** if we add a non-edge to $\bar{M}$ when iterating c in Algorithm 4; as such, $|\bar{M}|$ is equal to the number of successful colors.

We first show that for most colors, $\mathsf{pre}(c)$ is sufficiently large. We say a color c is **heavy** iff $\mathsf{pre}(c) \geqslant t/2$ and otherwise it is **light**. We first prove that at least half the colors are heavy.

Claim 16. *At any point in Algorithm 4, we have at least $\Delta/2$ heavy colors.*

Proof. For $\bar{f} = (u, v)$, let $B(\bar{f})$ be the set of colors used in the neighborhood of either u or v outside C. Since out-degree of any vertex in C is at most $10\varepsilon\Delta$ by Proposition 10, $|B(\bar{f})| \leqslant 20\varepsilon\Delta$. As a result, at the beginning of the algorithm,

$$\sum_{\bar{f} \in \bar{F}} |B(\bar{f})| \leqslant t \cdot 20\varepsilon\Delta.$$

This means that an average color appears in $B(\bar{f})$ for at most $20\varepsilon t$ non-edges $\bar{f}$; an application of Markov bound now implies that for at least $\Delta/2$ colors c, the color $c \in B(\bar{f})$ for at most $40\varepsilon t$ non-edges $\bar{f}$. This implies that at least $\Delta/2$ colors are "present" in at least $t - 80\varepsilon t \geqslant 0.9 \cdot t$ non-edges.

Additionally, whenever we add a non-edge (u, v) to $\bar{M}$, we remove all non-edge neighbors of u, v from $\bar{F}$; considering the in-non-degree of any vertex in C is at most $10\varepsilon\Delta$ by Proposition 10, we may delete up to $\ell \cdot 20\varepsilon\Delta = (0.1) \cdot t$ non-edges from $\bar{F}$ ever. As such, for each of the $\Delta/2$ colors c found above, c can be used to color $\geqslant 0.9t - (0.1)t$ non-edges, thus $\mathsf{pre}(c) \geqslant t/2$ for these $\Delta/2$ colors, making them heavy. □ Claim 16

We now use the above claim to say that whenever we are processing a heavy color, there is a "good" probability that we add a new edge to $\bar{M}$.

Claim 17. *For any heavy color c, $\Pr(c \text{ is successful}) \geqslant 10^{-5} \cdot t/\varepsilon\Delta^2$.*

Proof. The color c is successful if at least one non-edge $(u, v) \in \mathsf{Present}(c)$ have $c \in L_i(u) \cap L_i(v)$. By a simple application of the inclusion-exclusion principle,

$$\Pr(c \text{ is successful}) \geqslant \Pr\left(\text{at least one } (u, v) \in \mathsf{Present}(c) \text{ have } c \in L_i(u) \cap L_i(v)\right)$$

$$\geqslant \sum_{\substack{(u,v)\in \\ \mathsf{Present}(c)}} \Pr\left(c \in L_i(u) \cap L_i(v)\right) - \sum_{\substack{(u,v),(u',v')\in \\ \mathsf{Present}(c)}} \Pr\left(c \in L_i(u) \cap L_i(v) \cap L_i(u') \cap L_i(v')\right). \tag{3}$$

Let q be the probability that $c \in L(w)$ for any $w \in V$ (the probabilities are all equal to each other). The first term above is easy to bound as each c belongs to $L_i(u) \cap L_i(v)$ w.p. q^2, by the independence between the choice of lists. Hence,

$$\sum_{\substack{(u,v)\in \\ \mathsf{Present}(c)}} \Pr\left(c \in L_i(u) \cap L_i(v)\right) = \mathsf{pre}(c) \cdot q^2.$$

For the second term of Eq (3), there are two types of pairs of (distinct) non-edges $(u,v),(u',v')$ that we need to take into account: the ones that share exactly one endpoint and the ones that do not share any. There are at most $\mathsf{pre}(c) \cdot 10\varepsilon\Delta$ many non-edges of the first type as maximum non-degree of any vertex is at most $10\varepsilon\Delta$ by Proposition 10; there are also at most $\mathsf{pre}(c)^2$ many non-edges of the second type. Hence,

$$\sum_{\substack{(u,v),(u',v')\in \\ \mathsf{Present}(c)}} \Pr\left(c \in L_i(u) \cap L_i(v) \cap L_i(u') \cap L_i(v')\right) \leqslant \mathsf{pre}(c) \cdot 10\varepsilon\Delta \cdot q^3 + \mathsf{pre}(c)^2 \cdot q^4.$$

Finally, note that $q = 1$

We can plugin the above two bounds in Eq (3), and have,

$$\begin{aligned}
\Pr\left(c \text{ is successful}\right) &\geqslant \mathsf{pre}(c) \cdot q^2 - \mathsf{pre}(c) \cdot 10\varepsilon\Delta \cdot q^3 - \mathsf{pre}(c)^2 \cdot q^4 \\
&\geqslant (0.9) \cdot \mathsf{pre}(c) \cdot q^2 - \mathsf{pre}(c)^2 \cdot q^4 \\
&\qquad\qquad (\text{as } q = 1/(100\sqrt{\varepsilon}\Delta) \text{ and thus } 10\varepsilon\Delta \cdot q < (0.1) \text{ for } \varepsilon < 1) \\
&= (0.9) \cdot \mathsf{pre}(c) \cdot q^2 \cdot (1 - 20\varepsilon\Delta^2 \cdot q^2) \\
&\qquad\qquad (\text{as } \mathsf{pre}(c) \leqslant |\bar{F}| \leqslant 20\varepsilon\Delta^2 \text{ by the decomposition of Proposition 10}) \\
&\geqslant (0.8) \cdot \mathsf{pre}(c) \cdot q^2 \qquad (\text{as } q = (1/100\sqrt{\varepsilon}\Delta) \text{ and thus } (1 - 20\varepsilon\Delta^2 \cdot q^2 = 0.998)) \\
&\geqslant \frac{t}{10^5 \cdot \varepsilon\Delta^2}.
\end{aligned}$$

as desired. □ Claim 17

We are now ready to conclude the proof of Lemma 15. Let $\theta := 10^{-5} \cdot t/\varepsilon\Delta^2$ (the RHS of Claim 17); note that $\theta < 1$ because $t \leqslant |\bar{F}| \leqslant 20\varepsilon\Delta^2$ by the decomposition of Proposition 10. Let Z be a random variable sampled from binomial distribution $\mathcal{B}(\Delta/2, \theta)$. By Claim 17 and the fact that there are $\Delta/2$ heavy colors, plus a straightforward coupling argument, for every $\ell' \leqslant \ell$:

$$\Pr\left(|\bar{M}| \geqslant \ell'\right) \geqslant \Pr\left(Z \geqslant \ell'\right). \tag{4}$$

On the other hand, $\mathbb{E}[Z] = \Delta/2 \cdot \theta = 10^{-5} \cdot t/2\varepsilon\Delta$. By Chernoff bound (Proposition 4),

$$\begin{aligned}
\Pr\left(Z < \frac{1}{2} \cdot \mathbb{E}[Z]\right) &\leqslant \exp\left(-\frac{\mathbb{E}[Z]}{12}\right) \\
&= \exp\left(-\frac{t}{24 \cdot 10^5 \cdot \varepsilon\Delta}\right) \leqslant \exp\left(-\frac{10^7 \cdot \varepsilon\Delta}{24 \cdot 10^5 \cdot \varepsilon\Delta}\right) \qquad (\text{as } t \geqslant 10^7 \cdot \varepsilon\Delta) \\
&= \exp(-4) < 1/2.
\end{aligned}$$

As such, with probability at least 1/2,

$$Z \geqslant 10^{-5} \cdot t/4\varepsilon\Delta > 10^{-6} \cdot t/\varepsilon\Delta.$$

proving the lemma by Eq (4). □

Now, considering we can run Algorithm 4 for $\Theta(\log n)$ choices of lists $\{L_i(v) \mid v \in V\}$, we obtain that with high probability, there is a colorful non-edge matching $\bar{M}$ of size $\ell = t/\varepsilon\Delta$ in the almost-clique C. We can thus **color** the vertices of $\bar{M}$ accordingly, and by Eq (2), obtain the following lemma as a direct corollary.

Lemma 18. *Suppose the number of non-edges of C is $t \geqslant 10^7 \cdot \varepsilon\Delta$. Then, with high probability, we can color a subset of vertices in C without using $\{L^*(v) \mid v \in C\}$ such that:*

$$|\mathsf{Remain}(C)| = |C| - 2 \cdot \frac{t}{10^6 \cdot \varepsilon\Delta} \qquad , \qquad |\mathsf{Colors}(C)| = \Delta + 1 - \frac{t}{10^6 \cdot \varepsilon\Delta},$$

where $\mathsf{Remain}(C)$ is the set of uncolored vertices in C and $\mathsf{Colors}(C)$ is the set of colors not used in C.

This concludes the first part of coloring the almost-clique C. We emphasize that we only run Algorithm 4 when the preconditions of Lemma 18 are satisfied and otherwise directly go to the next step.

Part Two-(B): Final Coloring of the Almost-Clique C

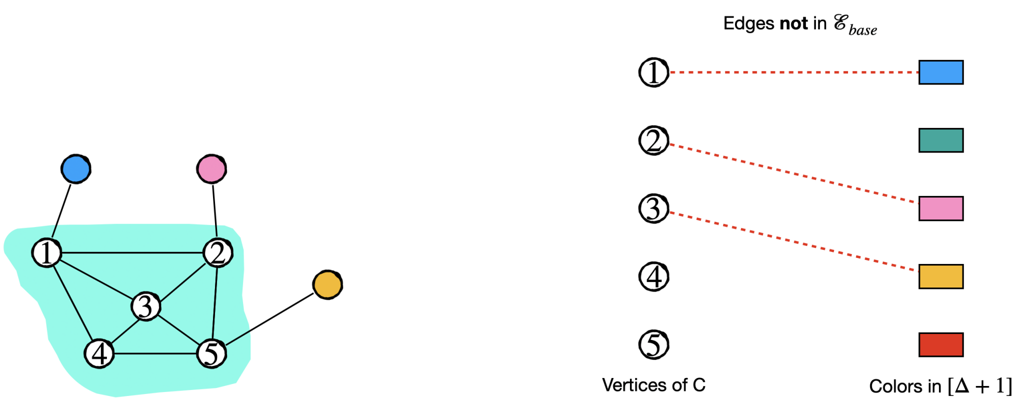
The final part of coloring the almost-clique C is to use a similar random graph theory type argument in the spirit of the case for cliques in Lemma 7. Define the following bipartite graph $\mathcal{G}_{base} := (\mathcal{L}, \mathcal{R}, \mathcal{E}_{base})$, called the *base palette graph*, for the almost-clique C:

- $\mathcal{L} := \mathsf{Remain}(C)$, i.e., the set of vertices yet to be colored in C; we will use vertices of $\mathsf{Remain}(C)$ and vertices of $\mathcal{L}$ interchangeably;
- $\mathcal{R} := \mathsf{Colors}(C)$, i.e., the set of colors that are yet to be used inside C; we will use colors of $\mathsf{Colors}(C)$ and vertices of $\mathcal{R}$ interchangeably;
- $\mathcal{E}_{base}$: any vertex $v \in \mathcal{L}$ has an edge to any color $c \in \mathcal{R}$ iff c is not used to color any neighbor of v in G outside C.

Similarly, define the **palette graph** of the almost-clique C as the bipartite graph $\mathcal{G} := (\mathcal{L}, \mathcal{R}, \mathcal{E})$ on the same set of vertices as $\mathcal{G}_{base}$ with the following edges:

- $\mathcal{E} \subseteq \mathcal{E}_{base}$: we only include an edge $(v, c) \in \mathcal{E}_{base}$ inside $\mathcal{E}$ as well if $c \in L^*(v)$.

As a result, $\mathcal{G}$ is a subgraph of $\mathcal{G}_{base}$ obtained by sampling edges via the lists $\{L^*(v) \mid v \in C\}$.



(a) An almost-clique with outside colors. (b) The (complement) of the base palette graph.

Figure 5: An illustration of the base palette graph for the almost-clique in (a) —we note that in figure (b), the depicted edges are the ones missing from the base palette graph.

In the following, we shall list several key properties of the base palette graph. To start, let us show that $\mathcal{L}$ is always at most as large as $\mathcal{R}$ or alternatively, the number of remaining vertices to be colored is at most the number of available colors.

Claim 19. *In $\mathcal{G}_{base}$ and $\mathcal{G}$, $|\mathcal{L}| \leqslant |\mathcal{R}|$.*

Proof. Suppose first that the number of non-edges t in C is $< 10^7 \cdot \varepsilon\Delta$ which is $< \Delta$. Then, $|\mathsf{Remain}(C)| = |C| \leqslant (\Delta+1)$ (because when $|C| > \Delta+1$, we right away have $t > \Delta$ by just the bound on maximum degree) and $|\mathsf{Colors}(C)| = \Delta + 1$ (because we do not even run Algorithm 4 in this case). Thus, $|\mathcal{L}| \leqslant |\mathcal{R}|$.

Now suppose $|C| = \Delta + 1 + k$ for some $k > 1$ which necessarily implies $t \geqslant \Delta > 10^7 \cdot \varepsilon\Delta$ and thus we know that Algorithm 4 was used in this case. We have $t \geqslant k \cdot \Delta$, which, by Lemma 18, implies that we will find a non-edge matching of size $10^{-6} \cdot k \cdot \Delta/\varepsilon\Delta = 10^{-6} \cdot k/\varepsilon > 2k$. As a result, we have,

$$|\mathcal{L}| = |\mathsf{Remain}(C)| = \Delta + 1 + k - 4k = \Delta + 1 - 3k \qquad \text{and} \qquad |\mathcal{R}| = |\mathsf{Colors}(C)| = \Delta + 1 - 2k,$$

which concludes the proof. □

The fact that $|\mathcal{L}| \leqslant |\mathcal{R}|$ suggests that that possibility of finding a ***$\mathcal{L}$*-perfect matching** $\mathcal{M}$ in $\mathcal{G}$ that matches all vertices of $\mathcal{L}$ is not entirely out of the question. Assuming such a matching exists, we will be done as in Lemma 7: we can color each vertex v of C with the color of $\mathcal{M}(v)$ and obtain a coloring of C from lists $\{L^*(v) \mid v \in C\}$ (by definition of $\mathcal{G}$) which is consistent with outside coloring (by definition of $\mathcal{G}_{base}$).

Of course, the fact that $|\mathcal{L}| \leqslant |\mathcal{R}|$ by no means implies that $\mathcal{G}$ (or even $\mathcal{G}_{base}$) has a $\mathcal{L}$-perfect matching. In the following, we list further properties of $\mathcal{G}_{base}$, which follow from the decomposition in Proposition 10 and the preprocessing step in Lemma 18. We then use these to prove the existence of the required matching.

Properties of $\mathcal{G}_{base}$. Let m denote the number of vertices in $\mathcal{L}$, i.e. $m := |\mathcal{L}|$ in $\mathcal{G}_{base}$. Conditioned on the high probability event of Lemma 18, $\mathcal{G}_{base}$ satisfies the following properties:

(i) $m \geqslant (2/3) \cdot \Delta$;

Proof: the maximum value of t, the number of non-edges in C is at most $20\varepsilon\Delta^2$ by Proposition 10. As such, the minimum possible size of C is at least $(1 - 5\varepsilon)\Delta - \frac{40\varepsilon\Delta^2}{10^6 \cdot \varepsilon\Delta}$ which is at least $(2/3) \cdot \Delta$.

(ii) $m \leqslant |\mathcal{R}| \leqslant 2m$;

Proof: by part (i) and the fact that $|\mathcal{R}| \leqslant \Delta + 1$.

(iii) minimum degree of vertices in $\mathcal{L}$ is at least $2m/3$, i.e., $\min_{v \in \mathcal{L}} \deg_{\mathcal{G}_{base}}(v) \geqslant 2m/3$.

Proof: any vertex $v \in \mathsf{Remain}(C)$ has up to $10\varepsilon\Delta$ edges out of C by Proposition 10 which implies that up to $10\varepsilon\Delta$ colors in $\mathcal{R}$ may be "blocked" for v meaning that v has no edges to them in $\mathcal{R}$. All the other colors are available to v making the degree of v at least $|\mathcal{R}| - 10\varepsilon\Delta > 2m/3$ by parts $(i), (ii)$.

(iv) for any vertex $v \in \mathcal{L}$,

$$\deg_{\mathcal{G}_{base}}(v) \geqslant |\mathcal{R}| - (\Delta + 1) + |C| - \overline{\deg}_C(v),$$

where $\overline{\deg}_C(v)$ is the number of non-neighbors of v in C;

Proof: $\deg_{\mathcal{G}_{base}}(v)$ is equal to the number of colors in $\mathsf{Colors}(c)$ that are not blocked for v, i.e., do not appear in the neighborhood of v outside C. Thus, we only need to count the number of neighbors of v outside C which are most $\Delta - (|C| - 1 - \overline{\deg}_C(v))$ as degree of v is at most Δ and it is neighbor to all but $\overline{\deg}_C(v) + 1$ vertices in C (including itself). Thus, $|\mathsf{Colors}(C)|$ minus these many colors are not blocked for v, implying the result.

(v) for any set $S \subseteq \mathcal{L}$ of size $|S| \geqslant m/2$,

$$\sum_{v \in S} \deg_{\mathcal{G}_{base}}(v) \geqslant (|S| \cdot m) - m/4.$$[11]

Proof: Suppose first that $t < 10^7 \cdot \varepsilon\Delta$ and thus $|\mathcal{L}| = |C|$ and $|\mathcal{R}| = \Delta+1$ (as also argued in Claim 19). By part (iv), and since $\sum_{v \in S} \overline{\deg}_C(v) \leqslant 2t$, we can write:

$$\sum_{v \in S} \deg_{\mathcal{G}_{base}}(v) \geqslant \sum_{v \in S} \left(|\mathcal{L}| - \overline{\deg}_C(v)\right) \geqslant |S| \cdot m - 2t \geqslant |S| \cdot m - 10^7 \cdot 2\varepsilon\Delta \geqslant |S| \cdot m - m/4,$$

as $\Delta \leqslant (3/2)m$ by part (i).

Now, consider the other case when $t \geqslant 10^7 \cdot \varepsilon\Delta$ and let $T := 10^{-6} \cdot t/\varepsilon\Delta$. Thus $|\mathcal{L}| = |C| - 2 \cdot T$ and $|\mathcal{R}| = \Delta + 1 - T$ by Lemma 18. By part (iv),

$$\begin{aligned}\sum_{v \in S} \deg_{\mathcal{G}_{base}}(v) &\geqslant \sum_{v \in S} (\underbrace{\Delta + 1 - T}_{|\mathcal{R}|} - (\Delta + 1) + \underbrace{|\mathcal{L}| + 2T}_{|C|} - \overline{\deg}_C(v)) \\ &\geqslant |S| \cdot |\mathcal{L}| + |S| \cdot T - 2t \geqslant |S| \cdot m + (2/3) \cdot \Delta \cdot 10^{-6} \cdot t/\varepsilon\Delta - 2t > |S| \cdot m,\end{aligned}$$

by the choice of T and part (i) for the second to last inequality.

The above are all the properties of $\mathcal{G}_{base}$ we need for the rest of the proof, i.e., to show that a random subgraph $\mathcal{G}$ of $\mathcal{G}_{base}$ wherein each edge is sampled w.p. q has a $\mathcal{L}$-perfect matching. As a warm-up, let us first quickly argue that $\mathcal{G}_{base}$ itself has a $\mathcal{L}$-perfect matching (we actually do not need this claim for the proof but it will provide a basic intuition).

Claim 20. *Any graph $\mathcal{G}_{base}$ satisfying the properties (i) to (v) above has a $\mathcal{L}$-perfect matching.*

Proof. By Hall's theorem, we only need to show that any set $S \subseteq \mathcal{L}$ has $|N_{\mathcal{G}_{base}}(S)| \geqslant |S|$. This is true for any set S of $\leqslant m/2$ as by property (iii), the minimum degree of any vertex in S is already $2m/3 > |S|$. For any set S of size $\geqslant m/2$, by property (v), there are at least $|S|m - m/4$ edges going out of S. As each vertex in $\mathcal{R}$ can have at most $|S|$ edges to S, there should be at least m neighbors for S in $\mathcal{R}$ as otherwise the total number of edges between the two sets would be $(m-1) \cdot |S| < |S|m - m/4$. As such, by Hall's theorem, $\mathcal{G}_{base}$ has a $\mathcal{L}$-perfect matching. □

It is worth pointing out that for Claim 20 to hold, we need the RHS of property (v) to be strictly larger than $|S|m - m$ (which is indeed true as it is $|S|m - m/4$). This is a necessary condition as otherwise $\mathcal{G}_{base}$ maybe a bipartite clique between $\mathcal{L}$ and $m-1$ vertices in $\mathcal{R}$ hence not having any $\mathcal{L}$-perfect matching.

The very last step of the proof is now to show that random subgraphs of $\mathcal{G}_{base}$ (or rather any graph satisfying properties (i) to (v)) also have a perfect matching with high probability. This can be seen as some form of generalization of typical random graph theory arguments. In (truly) random graphs, we work with random subgraphs of a clique (or a bipartite clique)—here instead, our "base" graph $\mathcal{G}_{base}$ from which the sampling is done is not a clique any more (and in fact can have up to $\Omega(m^2)$ edges missing from a (lopsided) bipartite clique). As such, the argument needs to take care of these differences explicitly.

In the following lemma, we slightly abuse the notation and change the model: instead of assuming the lists $L^*(v)$ are $\Theta(\log n)$ sampled colors from $[\Delta+1]$, we take each $L^*(v)$ to be a set of colors obtained by picking each color in $[\Delta+1]$ independently and with probability $\Theta(\log(n)/\Delta)$; this is however without loss of generality as with high probability, the distribution of sampled colors in this case also only samples $\Theta(\log n)$ colors as is required in the original model.

[11] Notice that the guarantee provided by this property is much stronger than applying property (ii) to each vertex of S and this is crucial for the proof; see Claim 20 for the importance of this property.

Lemma 21. *The subgraph $\mathcal{G}$ of $\mathcal{G}_{base}$ obtained by sampling each edge of $\mathcal{G}_{base}$ independently and with probability $p := \frac{100 \ln n}{m}$ (corresponding to (v, c) pairs where $c \in L(c)$ and since $m = \Theta(\Delta)$), with high probability has a $\mathcal{L}$-perfect matching.*

Proof. Similar to the proof of Lemma 7 (and Claim 20), we only need to show that for every set $S \subseteq \mathcal{L}$, $|N_{\mathcal{G}}(S)| \geqslant |S|$. We prove this in the following by considering two different cases based on the size of S.

Case 1: When S is "small", i.e., $|S| \leqslant m/2$. In this case, we can simply use the fact that minimum degree of $\mathcal{L}$-vertices in $\mathcal{G}_{base}$ is quite large to argue that $|N_{\mathcal{G}}(S)|$ should be large also. The proof of this part is almost identical to that of Lemma 7. Let us formalize this as follows.

Fix a set $S \subseteq \mathcal{L}$ of size at most $m/2$ and a set $T \subseteq \mathcal{R}$ of size $|S| - 1$. By property (iii) on the min-degree of vertices in $S \subseteq \mathcal{L}$, there at at least $|S| \cdot (2m/3 - |T|) \geqslant |S| \cdot m/6$ edges that are going from S to outside of T in $\mathcal{G}_{base}$. As each of these edges appears in $\mathcal{G}$ with probability $p = 100 \cdot \ln n/m$ independently, we have,

$$\Pr\left(N_{\mathcal{G}}(S) \subseteq T\right) \leqslant (1-p)^{|S| \cdot (m/6)} \leqslant \exp\left(-|S| \cdot (m/6) \cdot 100 \cdot \ln n/m\right) \leqslant n^{-10|S|}.$$

We can now do a union bound over all choices of S, T and have that,

$$\Pr\left(\text{there is a set } S \subseteq \mathcal{L} \text{ of size} \leqslant m/2 \text{ with } |N_{\mathcal{G}}(S)| < |S|\right) \leqslant \sum_{k=1}^{m/2} \binom{m}{k}^2 \cdot n^{-10k} \leqslant n^{-9},$$

as $m \leqslant n$. Thus, we proved that with high probability there is no Hall's witness set S with $|S| \leqslant m/2$.

Case 2: When S is "large", i.e., $|S| > m/2$. This is the main part of the proof. For this case, we can no longer only rely on the property (ii) of $\mathcal{G}_{base}$ and instead have to use property (v) crucially.

For the rest of the proof, let us set up the following notation. Fix a set S of size $\ell \geqslant m/2$. By property (v) of $\mathcal{G}_{base}$, there are at least $\ell \cdot m - m/4$ edges incident on S. Let us define the following:

- We pick a exactly $\ell \cdot m - m/4$ edges incident on S and denote them by E_S;
- For any vertex $c_i \in \mathcal{R}$, we define d_i as the degree of c_i in E_S;
- For any vertex $c_i \in \mathcal{R}$, we define an indicator random variable X_i where $X_i = 1$ iff c_i does not belong to $N_{\mathcal{G}}(S)$ and otherwise $X_i = 0$. Finally, define $X := \sum_i X_i$.

Note that by this definition X denotes the number of vertices in $\mathcal{R}$ not in $N_{\mathcal{G}}(S)$ and so $|N_{\mathcal{G}}(S)| = r - X$. As such, we are interested in bounding the probability:

$$\Pr\left(|N_{\mathcal{G}}(S)| < |S|\right) = \Pr\left(X > r - \ell\right). \tag{5}$$

Moreover, by their definition and the definition of $\mathcal{G}$ as a random subgraph of $\mathcal{G}_{base}$, the random variables $X_1, \ldots, X_r$ and corresponding $d_1, \ldots, d_r$ satisfy the following properties:

$$\begin{aligned}
&X_1, \ldots, X_r \text{ are independent;} \\
&\text{for } i \in [r] \qquad \Pr\left(X_i = 1\right) = (1-p)^{d_i}; \\
&\text{for } i \in [r] \qquad d_i \leqslant \ell; \\
&\text{and} \qquad \sum_{i=1}^{r} d_i = (\ell \cdot m) - m/4.
\end{aligned} \tag{6}$$

It turns out bounding these probabilities directly is rather hard. Instead, in the following, we first show the "worst case" possible values for these variables (corresponding to a natural worst case graph) and then bound the probability of Eq (5) under these variables. Formally,

Claim 22. *Consider the following program on* <u>*independent*</u> *random variables* $Y_1, \ldots, Y_r$*:*

$$\textit{maximize}_{\{z_i\}_{i=1}^r} \Pr\left(Y := \sum_i Y_i > r - \ell\right) \quad \textit{subject to}$$

$$\textit{for any } i \in [r] \qquad \Pr(Y_i = 1) = (1-p)^{z_i} \quad \textit{and} \quad 0 \leqslant z_i \leqslant \ell; \qquad \sum_{i=1}^r z_i = (\ell \cdot m) - m/4;$$

(We emphasize that the only variables of this program are the integers $(z_1, \ldots, z_r)$*).*

Then, an optimal solution is $(z_1^*, \ldots, z_r^*) = (\underbrace{\ell, \ell, \cdots, \ell}_{m-1}, \ell - m/4, 0, \ldots, 0)$.

Before getting to the proof of this claim, let us provide some further intuition. Consider the case when the base graph $\mathcal{G}_{base}$ is such that there are $m-1$ vertices with degree ℓ to S, one vertex with degree $\ell - m/4$, and all other vertices in $\mathcal{R}$ have degree zero to $\mathcal{R}$ (such a choice satisfies Eq (6)). Then, the random variables $Y_1, \ldots, Y_r$ and variables $z_1^*, \ldots, z_r^*$ in Claim 22 would correspond to random variables $X_1, \ldots, X_r$ and $d_1, \ldots, d_r$ in Eq (6). By Eq (5) and the optimality of $z_1^*, \ldots, z_r^*$, we obtain that $\Pr(|N_{\mathcal{G}}(S)| < |S|)$ is maximized under such a graph, turning it a worst-case graph for proving the lemma in Case 2.

Another good intuition here is that in defining this worst-case graph, we simply moved degrees of all vertices in $\mathcal{R}$ to a *minimal* set of m vertices while satisfying the degree requirements of Eq (6); one may expect this to be the worst-case example as any deviation from this can only increase the chance of another vertex also joining $N_{\mathcal{G}}(S)$ while decreasing the chance of an original vertices to be in $N_{\mathcal{G}}(S)$ by a *lower* amount[12].

Proof of Claim 22. Consider any solution $(z_1, \ldots, z_r)$ to the program. We show that we can transform the solution one variable at a time toward $(z_1^*, \ldots, z_r^*)$, without decreasing the value of the objective function. This will imply the optimality of $(z_1^*, \ldots, z_r^*)$.

Without loss of generality assume $z_1 \geqslant z_2 \geqslant \cdots \geqslant z_r$ (as the objective function is symmetric in the variables). If $(z_1, \ldots, z_r) \neq (z_1^*, \ldots, z_r^*)$, then there is an index i such that $z_i < z_i^*$. Consider the intermediate solution[13]:

$$(z_1', \ldots, z_r') = (z_1, \ldots, z_{i-1}, z_i + 1, z_{i+1} - 1, z_{i+2}, \ldots, z_r),$$

which is a feasible solution because $z_i < z_i^* \leqslant \ell$ and $z_{i+1} > 0$ as $\sum_i z_i = \sum_i z_i^*$ and both sequences are sorted. We now prove that this change of variables increase the value of the objective function. The change above only affects the random variables Y_i and Y_{i+1} in the objective function. As such, we only need to prove that for any $t \in \{1, 2\}$,

$$\Pr\left(Y_i + Y_{i+1} = t \ \text{ with variables } z_i', z_{i+1}'\right) \geqslant \Pr\left(Y_i + Y_{i+1} = t \ \text{ with variables } z_i, z_{i+1}\right).$$

For $t = 1$, this is true because:

$$\begin{aligned}
\Pr\left(Y_i + Y_{i+1} = 1 \ \text{ with variables } z_i', z_{i+1}'\right) &= (1-p)^{z_i'} + (1-p)^{z_{i+1}'} - (1-p)^{z_i' + z_{i+1}'} \\
&= (1-p)^{z_i+1} + (1-p)^{z_{i+1}-1} - (1-p)^{z_i + z_{i+1}} \\
&= (1-p) \cdot (1-p)^{z_i} + (1-p)^{-1} \cdot (1-p)^{z_{i+1}} - (1-p)^{z_i + z_{i+1}} \\
&\geqslant (1-p)^{z_i} + (1-p)^{z_{i+1}} - (1-p)^{z_i + z_{i+1}} \\
\text{(as } (1-p)^{z_i} \leqslant (1-p)^{z_{i+1}} &\text{ and } \alpha \cdot x + \tfrac{1}{\alpha} \cdot y \geqslant x + y \text{ for } \alpha \leqslant 1 \text{ and } x \leqslant y) \\
&= \Pr\left(Y_i + Y_{i+1} = 1 \ \text{ with variables } z_i, z_{i+1}\right).
\end{aligned}$$

[12] We note that all this is just for the intuition—after proving Claim 22, we can directly use the answer to upper bound Eq (6) as the variables $X_1, \ldots, X_r$ and $d_1, \ldots, d_r$ form a valid solution to the program of Claim 22.

[13] In the context of the worst-case graph intuition, here, we are moving one edge of the base graph from a lower degree vertex v_{i+1} to a higher degree vertex v_i without violating any constraint in Eq (6).

For $t = 2$, this is true because:

$$\begin{aligned}\Pr\left(Y_i + Y_{i+1} = 2 \text{ with variables } z'_i, z'_{i+1}\right) &= (1-p)^{z'_i + z'_{i+1}} \\ &= (1-p)^{z_i + z_{i+1}} \\ &= \Pr\left(Y_i + Y_{i+1} = 2 \text{ with variables } z_i, z_{i+1}\right).\end{aligned}$$

As such, the value of the objective function does not increase by this change of variables. We can thus repeatedly apply such transformation to turn the original variables into $(z_1^*, \ldots, z_r^*)$ without ever decreasing the objective value, implying the optimality of the latter solution. $\square$ Claim 22

To upper bound the probability in Eq (5), we can instead upper bound the probability of the objective value of the program in Claim 22 with the assignment $(z_1^*, \ldots, z_r^*)$. Notice that under this assignment, $Y_{m+1} = Y_{m+2} = \ldots = Y_r = 1$ already as $z_{m+1}^* = \ldots = z_r^* = 0$, and so there is nothing to do for those variables. As such, our goal is to bound the probability

$$\Pr\left(Y := \sum_{i=1}^{r} Y_i > r - \ell\right) = \Pr\left(\tilde{Y} := \sum_{i=1}^{m} Y_i > m - \ell\right). \tag{7}$$

Now, fix a set $T \subseteq [m]$ of size $m - \ell + 1$. For $\tilde{Y} > m - \ell$ to be true, there should exists at least one T such that $Y_i = 1$ for all $i \in T$. In the following, we bound this probability for every single T and then do a union bound on all T. By the independence of variable Y_i's and the values of their marginals under $(z_1^*, \ldots, z_r^*)$ in Claim 22, we have,

$$\begin{aligned}\Pr\left(\text{for all } i \in T,\, Y_i = 1\right) &= \prod_{i \in T} \Pr\left(Y_i = 1\right) = (1-p)^{\sum_{i\in T} z_i^*} \leqslant (1-p)^{|T|\cdot \ell - m/4} \\ &\qquad\qquad \text{(as at most one } z_i^* = \ell - m/4 \text{ and the rest are } \ell) \\ &\leqslant \exp\left(-\frac{100 \ln n}{m} \cdot ((m - \ell + 1)\cdot \ell - m/4)\right) \\ &\qquad\qquad \text{(by the choice of } p = \tfrac{100 \ln n}{m} \text{ and } |T| = m - \ell + 1) \\ &\leqslant \exp\left(-\frac{100 \ln n}{m} \cdot ((m - \ell)\cdot \ell + m/4)\right) \qquad \text{(as } \ell = |S| > m/2 \text{ in Case 2)} \\ &\leqslant \exp\left(-25 \ln n \cdot (m - \ell + 1)\right) \qquad \text{(again as } \ell > m/2) \\ &= n^{-25\cdot(m-\ell+1)}.\end{aligned}$$

Applying a union bound over all choices of $T \subseteq [m]$ and Eq (7) implies that:

$$\Pr\left(Y := \sum_{i=1}^{r} Y_i > r - \ell\right) \leqslant \binom{m}{m - \ell + 1} \cdot n^{-25\cdot(m-\ell+1)} \leqslant n^{-24\cdot(m-\ell+1)},$$

as $m \leqslant n$. By Plugging in this inside Eq (5) (by the worst-case guarantee provided by Claim 22), we obtain

$$\Pr\left(|N_{\mathcal{G}}(S)| < |S)|\right) \leqslant n^{-24\cdot(m-|S|+1)}.$$

Finally, we can do a union bound over all sets $S \subseteq \mathcal{L}$ of size $\geqslant m/2$ and obtain that

$$\begin{aligned}\Pr\left(\text{there is a set } S \subseteq \mathcal{L} \text{ of size} > m/2 \text{ with } |N_{\mathcal{G}}(S)| < |S|\right) &\leqslant \sum_{k=m/2}^{m} \binom{m}{m-k} \cdot n^{-24\cdot(m-k+1)} \\ &\leqslant \sum_{k=m/2}^{m} n^{(m-k)} \cdot n^{-24\cdot(m-k+1)} \qquad \text{(as } m \leqslant n) \\ &\leqslant n^{-20}.\end{aligned}$$

As such in Case 2 also there is no Hall's theorem witness set in $\mathcal{G}$ with high probability. This implies that $\mathcal{G}$ has a $\mathcal{L}$-perfect matching with high probability, concluding the proof of the lemma. $\square$

By Lemma 21, there is a $\mathcal{L}$-perfect matching $\mathcal{M}$ in $\mathcal{G}$ with high probability and by the definition of the graph $\mathcal{G}$, we can use this matching to color any vertex $v \in \mathsf{Remain}(C)$ with the color $c = \mathcal{M}(v)$ which belongs to $L^*(v)$ (by the definition of $\mathcal{G}$) and also has not appeared in the neighborhood of v outside C (by the definition of $\mathcal{G}_{base}$) (and as $\mathcal{M}$ is a matching, c also does not appear elsewhere in C). This concludes Part 2-(B) of coloring almost-cliques and consequently the entire proof of Theorem 2.

Remark. As one can see, the majority of the arguments we used in the proof of Theorem 2 are algorithmic and constructive. In order to obtain the desired $\widetilde{O}(n^{3/2})$ *time* (and not only query) algorithm in Theorem 1, we only need to show that these algorithms can be implemented efficiently also, which is actually an easy task (and can be left to the reader to verify). However, there is still one more missing piece: how to find the sparse-dense decomposition? It turns out an "approximate" version of this decomposition can also be found using simple sampling algorithms (see [ACK19] or [AW22]) in $\widetilde{O}(n)$ time, which in turn gives us the final $\widetilde{O}(n^{3/2})$ time algorithm.

References

[ACK19] Sepehr Assadi, Yu Chen, and Sanjeev Khanna. Sublinear algorithms for $(\Delta + 1)$ vertex coloring. In *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2019, San Diego, California, USA, January 6-9, 2019*, pages 767–786, 2019. 2, 9, 23

[AW22] Sepehr Assadi and Chen Wang. Sublinear time and space algorithms for correlation clustering via sparse-dense decompositions. In Mark Braverman, editor, *13th Innovations in Theoretical Computer Science Conference, ITCS 2022, January 31 - February 3, 2022, Berkeley, CA, USA*, volume 215 of *LIPIcs*, pages 10:1–10:20. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022. 23

[Bol01] Béla Bollobás. *Random graphs*. Number 73. Cambridge university press, 2001. 5

[HSS16] David G Harris, Johannes Schneider, and Hsin-Hao Su. Distributed $(\Delta+1)$-coloring in sublogarithmic rounds. In *Proceedings of the forty-eighth annual ACM symposium on Theory of Computing*, pages 465–478. ACM, 2016. 9

[KK23] Jeff Kahn and Charles Kenney. Asymptotics for palette sparsification. *arXiv preprint arXiv:2306.00171*, 2023. 9

[MR97] Michael Molloy and Bruce Reed. A bound on the strong chromatic index of a graph. *Journal of Combinatorial Theory, Series B*, 69(2):103–109, 1997. 10

[MR02] Michael Molloy and Bruce Reed. **Graph coloring and the probabilistic method**. *New York I Springer*, 23:1329–356, 2002. 7, 11, 12

[Ree98] Bruce Reed. ω, δ, and χ. *Journal of Graph Theory*, 27(4):177–212, 1998. 7