CS 761: Randomized Algorithms **University of Waterloo: Winter 2025**

Homework 2

Due date: Thursday, March 12, 2025

Problem 1. Let us use information theory to prove the following inequality: for any $n \geqslant 1$ and $k \leqslant n/2$,

$$\sum_{i=0}^{k} \binom{n}{i} \leqslant 2^{n \cdot H_2(k/n)},$$

where for any $p \in (0,1)$, we define

$$H_2(p) := p \cdot \log\left(\frac{1}{p}\right) + (1-p) \cdot \log\left(\frac{1}{1-p}\right).$$

Note that for $p \leqslant 1/2$, we have $p \leqslant H_2(p)$.

Define $(X_1, X_2, \ldots, X_n)$ to be a random variable distributed uniformly over all n-bit strings with at most k ones. Prove that

$$\log\left(\sum_{i=0}^{k} \binom{n}{i}\right) = \mathbb{H}(X_1, \ldots, X_n) \leqslant n \cdot H_2(k/n),$$

which immediately implies the desired inequality. **(20 points)**

Problem 2. Suppose $M \in \{0,1\}^{m \times n}$ is a $m \times n$ dimensional matrix such that every row sums up to some integer r and every column sums up to some other integer c, i.e.,

$$\forall i \in [m]: \quad \sum_{j=1}^{n} M_{i,j} = r,$$

$$\forall j \in [n]: \quad \sum_{i=1}^{m} M_{i,j} = c.$$

Show that there exists a vector $X \in \{0,1\}^n$ such that for $Z = M \cdot X$, and any $i \in [m]$,

$$\frac{r}{2} - O(\sqrt{r \cdot \log(rc)}) \leqslant Z_i \leqslant \frac{r}{2} + O(\sqrt{r \cdot \log(rc)}).$$

(20 points)

Problem 3. For any $n \times n$ matrix A, the **Frobenius norm** of A is defined as

$$\|A\|_F := \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} (A_{i,j})^2}.$$

Let A and B be two $n \times n$ integer matrices. Recall that computing $A \cdot B$ takes $O(n^3)$ time using a direct algorithm and $O(n^{2.37\cdots})$ time using 'fast matrix multiplication'.

Design a randomized algorithm that for any $\varepsilon > 0$ in $O(n^2 \log(n)/\varepsilon^2)$ time, outputs a matrix C such that

$$\|AB - C\|_F \leqslant \varepsilon \cdot \|A\|_F \cdot \|B\|_F$$

with high probability. **(30 points)**

Hint: Recall JLL from Lecture 11 and consider computing $AS^\top SB$ for some appropriately chosen S.

Problem 4. Let $G = (V, E)$ be an undirected Δ-regular graph. For any parameter $\delta \in (0, 1)$, assume that:

- $\Delta \geqslant C \cdot \dfrac{\ln \Delta}{\delta^2}$ for some arbitrarily large constant $C > 0$;
- for every vertex $v \in V$, the total number of edges between the vertices in $N(v)$ is at most

$$\binom{\Delta}{2} - \delta \cdot \Delta^2.$$

Prove that G admits a vertex coloring with at most $(1 - \Theta(\delta)) \cdot \Delta$ colors. **(30 points)**

Hint: This question is related to the problem of coloring triangle-free graphs we studied in Lecture 10.

Problem 5 (Extra credit). Suppose we have an undirected graph $G = (V, E)$ with a list $L(v)$ of colors for each $v \in V$. For any vertex v and color c, we define the color-degree of (v, c) as the number of neighbors of v that contain the color c in their list, i.e.,

$$\deg(v, c) := |\, \{u \in N(v) \mid c \in L(u)\}\, |.$$

Let ℓ denote the minimum list size and d denote the maximum color-degree. Prove that if $\ell > 10 \cdot d$, then, there is a proper coloring of G such that each vertex $v \in V$ can be colored with a color from $L(v)$.

(+10 points)

Problem 6 (Extra credit). Prove that for any $\varepsilon > 0$ and any Δ sufficiently large as a function of ε, any Δ-regular triangle-free graph can be colored with $\varepsilon \cdot \Delta$ colors. **(+10 points)**