CS 671: Graph Streaming Algorithms and Lower Bounds **Rutgers: Fall 2020**

Problem set 8

Due: 11:59PM, November 3, 2020

Please solve **both** problems below.

Problem 1. Suppose $f : \{0,1\}^n \rightarrow \{0,1\}$ is a function and let μ be a distribution over $\{0,1\}^n$. We define the *average query complexity* of f over the distribution μ as:

$$D_\mu(f) := \min_{\text{algorithm } A \text{ that solves } f \text{ w.p. } \geq 2/3 \text{ over } x \sim \mu} \text{average number of queries of } A \text{ to } x,$$

where each query of A to $x \in \{0,1\}^n$ simply asks for the value of x_i for a given i; here, the average in the query complexity is taken over the choice of $x \sim \mu$. [1]

Define $f^m : (\{0,1\}^n) \rightarrow \{0,1\}^m$ where

$$f^m(x^1, \ldots, x^m) = (f(x^1), f(x^2), \ldots, f(x^m)).$$

The goal of this question is to prove a lower bound for $D_{\mu^m}(f^m)$ based on $D_\mu(f)$, i.e., a *direct sum* result for average query complexity of f. ($x^1, \ldots, x^m \sim \mu^m$ is sampled by picking each x^i independently from μ.) Formally, we like to prove that

$$D_{\mu^m}(f^m) \geq m \cdot D_\mu(f).$$

(*i*) Let A be any algorithm for f^m with probability of success $2/3$ and average query complexity q over μ. Define the following algorithm B for f over $x \sim \mu$:

- (a) Sample $i \in [m]$ uniformly at random and set $x^i = x$.
- (b) Sample $x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^m$ independently from μ.
- (c) *Simulate* running A over $(x^1, \ldots, x^m)$ and output the same answer that A outputs for x^i in f^m.

Show how to do the simulation and implement B in a way that it achieves a probability of success $2/3$ for f over μ, while having average query complexity q/m.

(*ii*) Use part (*i*) to prove the direct sum result.

Problem 2. Define Echo as the following communication problem: Alice gets a single bit $x \in \{0,1\}$ and Bob gets *no* input; the goal for Bob is to output x, i.e., "echo" x. Consider the distribution μ which is uniform over $\{0,1\}$. Clearly, Echo requires 1 bit of communication for $x \sim \mu$ to success with probability more than half.

(*i*) Prove that (external) information complexity of Echo over the distribution μ (with probability of success $2/3$) is also $\Omega(1)$.

Hint: Use Fano's inequality.

(*ii*) Use part (*i*) plus the direct sum of external information complexity for one-way protocol to prove that information complexity of the Index problem over uniform distribution on $\{0,1\}^n$ and $i \in [n]$ is $\Omega(n)$.

[1] It is easier to work with the average 'cost' of the algorithm in this problem compared to the typical worse-case cost. However, one can easily transition between the two by a small cost in query cost and probability of success of the algorithm.