CS 521: Linear Programming **Rutgers: Fall 2022**

Lecture 9

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Topics of this Lecture

1 Background on the Ellipsoid Method

1.1 Overview

We continue our study of cutting planes. In the last lecture, we saw the Center of Gravity method for convex optimization as well as LP feasibility. The center of gravity method was efficient in terms of the number of iterations it needed buts its main drawback was that computing the center of gravity of the search space is in general "hard" computationally. We are going to see another cutting plane, the *Ellipsoid method*, that remedies this drawback and results in the first polynomial time algorithm for solving LPs in this course[1].

The Ellipsoid algorithm, similar to the Center of Gravity method, is used for solving the LP feasibility problem: Given a polytope P in $\mathbb{R}^n$ characterized by m constraints $Ax \leqslant b$, decide whether or not P is empty, i.e., is feasible. Recall that the Center of Gravity method worked as follows:

(i) Start with an n-dimensional cube Q in $\mathbb{R}^n$ that contains the polytope P.

(ii) Compute the center of gravity $c \in \mathbb{R}^n$ of Q. Check if $c \in P$: if yes, return c is a certificate that P is feasible, otherwise let $a_i \in A$, $b_i \in b$ such that $\langle a_i\,,\,c\rangle > b_i$ be a violating constraint.

(iii) Update $Q \leftarrow Q \cap \{x \mid \langle a_i\,,\,x\rangle \leqslant \langle a_i\,,\,c\rangle\}$ and go back to the previous step.

Throughout this procedure, the polytope Q we maintain "traps" the polytope P until we eventually find a point in P itself or certify its infeasibility. The key challenge in this approach was that this trapping polytope

[1] Historically also, Ellipsoid is the first polynomial time algorithm for solving linear programs – you may want to check this article in this regard.

Q that we use in this method may become more and more complicated through the iterations, thus making computation of its center of gravity computationally hard.

The Ellipsoid method addresses this by changing the trapping polytope Q with another high-dimensional object whose center of gravity can be computed much simpler, namely, an *Ellipsoid* (a high-dimensional ellipse; we will define this formally shortly). At a high level, the new strategy in the Ellipsoid method is be as follows:

(*i*) Start with an n-dimensional ellipsoid Q in $\mathbb{R}^n$ that contains the polytope P.

(*ii*) Compute the center of gravity[2] $c \in \mathbb{R}^n$ of Q. Check if $c \in P$: if yes, return c is a certificate that P is feasible, otherwise let $a_i \in A$, $b_i \in b$ such that $\langle a_i \, , \, c \rangle > b_i$ be a violating constraint.

(*iii*) Update Q to be the minimum volume ellipsoid containing the half-ellipsoid $Q \cap \{x \mid \langle a_i \, , \, x \rangle \leqslant \langle a_i \, , \, c \rangle\}$ and go back to the previous step.

See Figure 1 for an illustration.

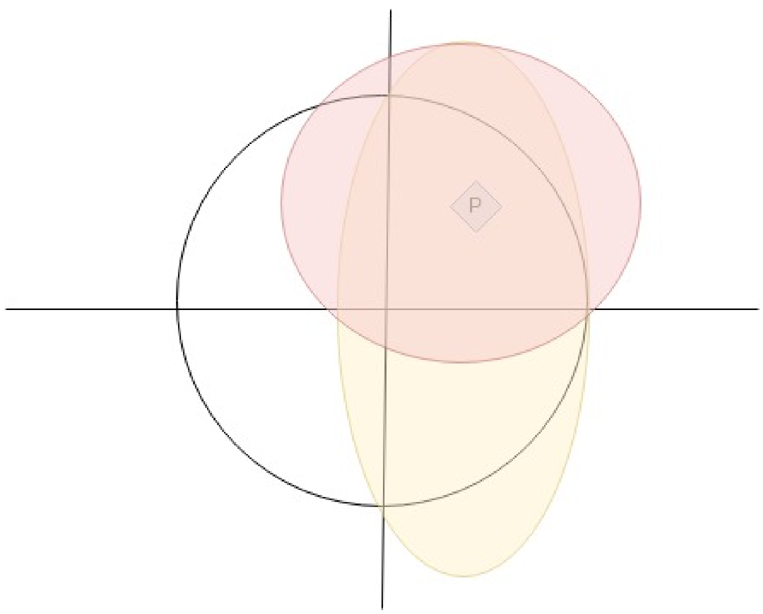

Figure 1: Graphical representation of three steps of the Ellipsoid method.

Two very important remarks are in order. Firstly, since we now aim to trap the half-ellipsoid $Q \cap \{x \mid \langle a_i \, , \, x \rangle \leqslant \langle a_i \, , \, c \rangle\}$ with another ellipsoid, it means that we are inevitably going to "bring in" some points beyond this half-ellipsoid into Q; in other words, Q may contain some points that we already ruled out as infeasible before, but we are bringing them back into Q again. This is necessary to maintain the fact that Q is an ellipsoid, which in turn is crucial for us to be able to compute its center of gravity efficiently. Secondly, and primarily because of the same reason, the volume of the ellipsoid Q does *not* drop by a constant factor (unlike the center of gravity method which drops the volume of the trapping polytope by a constant factor in each iteration). Instead, it drops only by a factor roughly $(1 - 1/n)$. Roughly speaking, each n iterations of the ellipsoid method are "as efficient" as just one iteration of the center of gravity. Thus, the ellipsoid method has an n^2-dependence in its number of iterations, i.e., quadratically worse than the center of gravity method; but now, each iteration can be implemented in polynomial time.

[2] As we shall see, the center of gravity of an ellipsoid can be obtained "for free" from the formulation of the ellipsoid we use in this algorithm.

1.2 Formal Definitions and Preliminary Tools

We will now start formalizing the above intuition into an actual algorithm. To do so, we need to present some mathematical background and the preliminary tools first. Throughout this lecture, unless specified otherwise, all the norms are ℓ_2-norms.

High-dimensional balls. We start by defining high-dimensional balls.

Definition 1. A **(closed) ball** $\mathbb{R}^n$ centered at a point $c \in \mathbb{R}^n$ with radius $r \in \mathbb{R}$, denoted by $B(c, r)$, is the set:

$$B(c, r) := \{x \in \mathbb{R}^n \mid \|x - c\|^2 \leqslant r^2\}.$$

Notice that the center of gravity of any ball $B(c, r)$ is simply the point c itself. The set $B(0, 1)$ is called the *unit ball* and is defined by $B(0, 1) := \{x \mid \|x\|^2 \leq 1\}$.

Ellipsoids. An ellipsoid is simply any affine transformation of the unit ball.

Definition 2. An **ellipsoid** E centered at the origin is the image $L(B(0, 1))$ of the unit ball under an *invertible* linear transformation $L : \mathbb{R}^n \to \mathbb{R}^n$. An ellipsoid centered at a general point $c \in \mathbb{R}^n$ is just the translation $c + E$ of some ellipsoid E centered at 0.

We now used this definition to obtain another formulation for Ellipsoids that can be more suitable to work in certain cases. Let A be the full-dimensional matrix in $\mathbb{R}^{n\times n}$ corresponding to the invertible linear transformation $L : \mathbb{R}^n \to \mathbb{R}^n$. We thus have,

$$\begin{aligned}
L(B(0,1)) &= \{y \mid A^{-1}y \in B(0,1)\} && \text{(by the definition of } L(B(0,1)) \text{ and since } L(x) = A \cdot x\text{)} \\
&= \{y \mid \|A^{-1}y\|^2 \leqslant 1\} && \text{(by the definition of } B(0,1)\text{)} \\
&= \{y \mid (A^{-1}y)^\top \cdot (A^{-1}y) \leqslant 1\} && \text{(as } \|z\| = z^\top \cdot z \text{ for any vector } z \in \mathbb{R}^n\text{)} \\
&= \{y \mid (y^\top {A^{-1}}^\top) \cdot (A^{-1}y) \leqslant 1\} && \text{(as } (Bz)^\top = z^\top B^\top \text{ for all } B \in \mathbb{R}^{n\times n} \text{ and } z \in \mathbb{R}^n\text{)} \\
&= \{y \mid y^\top (AA^\top)^{-1} y \leqslant 1\} && \text{(as } (B^{-1})^\top = (B^\top)^{-1} \text{ for any full-dimensional matrix } B \in \mathbb{R}^{n\times n}\text{)} \\
&= \{y \mid y^\top Q^{-1} y \leqslant 1\},
\end{aligned}$$

where we define Q to be the matrix $AA^\top$. The matrix Q is a *positive definite matrix* defined as follows.

Definition 3. A symmetric matrix $Q \in \mathbb{R}^{n\times n}$ is called **positive definite** iff the following equivalent conditions are true:

1. For all $x \neq 0 \in \mathbb{R}^n$, $x^\top Q x > 0$;
2. All eigenvalues of Q are real and positive.

As an aside, let us show that both these conditions are equivalent.

Proposition 4. *The two conditions in Definition 3 are equivalent.*

Proof. **(1. $\implies$ 2.)**. Let λ be an eigenvalue of Q with a corresponding eigenvector $v \in \mathbb{R}^n$. Then,

$$Q \cdot v = \lambda \cdot v \iff v^\top Q \cdot v = \lambda \cdot v^\top v = \lambda \cdot \|v\|^2.$$

By condition (1), we have that $v^\top Q \cdot v$ is positive, which implies that λ is also positive and real.

(2. $\implies$ 1.). By the eigendecomposition of Q and since Q is symmetric, we can factorize $Q = U \cdot D \cdot U^\top$ where D is a diagonal matrix with eigenvalues $(\lambda_1, \ldots, \lambda_n)$ of Q on the diagonal, For any $x \neq 0 \in \mathbb{R}^n$, define $y := U^\top x$; we have,

$$x^\top \cdot Q \cdot x = x^\top U \cdot D \cdot U^\top \cdot x = y^\top D \cdot y = \sum_{i=1}^{n} \lambda_i \cdot y_i^2 > 0,$$

since all $\lambda_i > 0$ for $i \in [n]$ by condition (2). ☐

Going back to the matrix $Q = AA^\top$ in the definition of $L(B(0,1))$, we have that for all $x \neq 0 \in \mathbb{R}^n$,

$$x^\top Q x = (x^\top A)(A^\top x) = \|Ax\|^2 > 0,$$

since A is full rank. Thus, Q is indeed a positive definite matrix.

Using this, we can alternatively define any ellipsoid as follows.

Definition 5. Given a symmetric positive definite matrix $Q \in \mathbb{R}^{n \times n}$, an ellipsoid centered at $c \in \mathbb{R}^n$ with respect to Q and c, denoted by $E(c, Q)$, can be defined as :

$$E(c, Q) = \{x \mid (x - c)^\top Q^{-1}(x - c) \leqslant 1\}. \tag{1}$$

As an example, the ball $B(0, r)$ can be written as the ellipsoid $E(0, r^2 \cdot I)$ where I is the identity matrix. This is because, we have,

$$E(0, r^2 \cdot I) = \left\{x \mid x^\top (r^2 \cdot I)^{-1} x \leqslant 1\right\} = \left\{x \mid x^\top \cdot x \leqslant r^2\right\} = \left\{x \mid \|x\|^2 \leqslant r^2\right\} = \{x \mid \|x\| \leqslant r\} = B(0, r).$$

The center of gravity of the Ellipsoid $E(c, Q)$ for any c is simply the point $c \in \mathbb{R}^n$.

Volume of ellipsoids. We obtain a simple formula specifying the volume of any ellipsoid. To do so, we need the following fact from linear algebra.

Fact 6. *Let $S \subseteq \mathbb{R}^n$ be any subset and $A \in \mathbb{R}^{n \times n}$ be a matrix. Define $A \cdot S := \{Ax \mid x \in S\}$. Then,*

$$\text{Volume}(A \cdot S) = |\det(A)| \cdot \text{Volume}(S).$$

Using this fact, we have the following formula for volume of ellipsoids.

Proposition 7. *For any ellipsoid $E(c, Q)$,* $\text{Volume}(E(c, Q)) = \sqrt{\det(Q)} \cdot \text{Volume}(B(0, 1))$.

Proof. We have $\text{Volume}(E(c, Q)) = \text{Volume}(E(0, Q))$ as the volume does not change by moving the center of the ellipsoid. Recall that any ellipsoid $E(0, Q)$ is the linear transformation $A \cdot B(0, 1)$ for some matrix A such that $Q = AA^\top$. Thus, by Fact 6,

$$\text{Volume}(E(c, Q)) = \text{Volume}(A \cdot B(0, 1)) = \det(A) \cdot \text{Volume}(B(0, 1)) = \sqrt{\det(Q)} \cdot \text{Volume}(B(0, 1)),$$

where in the last equality we used the fact that $\det(A) = \det(A^\top)$ and $\det(Q) = \det(A) \cdot \det(A^\top)$. ☐

Proposition 7 gives a simple way of computing the volume of an ellipsoid (on top of its center of gravity) which is needed for our analysis. Note that we could have also further specified the volume of the unit ball $B(0, 1)$, but it turns out that this quantity will be irrelevant for us as it cancels out in our calculations and we can simply consider it as our "unit of volume".

2 The Ellipsoid Algorithm

After having described our fundamental definitions and acquired familiarity with our mathematical elements, we now go on to describe the ellipsoid algorithm. Recall that the goal of the algorithm is to solve the feasibility problem for the polytope $P \subseteq \mathbb{R}^n$ described as $Ax \leqslant b$. Similar to the center of gravity algorithm, we make the following assumption[3].

Assumption 8. *The polytope (P) is a subset of the n-dimensional ball $B(0, R)$ for some $R \geqslant 1$. Moreover, if (P) is non-empty, then it fully contains an n-dimensional ball $B(c, r)$ for some $c \in \mathbb{R}^n$ and $r > 0$.*

We can now present the ellipsoid algorithm (modulo one of its key steps that will be specified later).

Ellipsoid Algorithm:

(*i*) We initialize the algorithm with the ellipsoid $E_1 = B(0, R) = E(c_1, Q_1)$, for $c_1 = 0$ and $Q_1 = r^2 \cdot I$.

(*ii*) While $\text{Volume}(E_t) > \text{Volume}(B(0, r))$:

  (a) If c_t is a feasible point in P, return c_t as a certificate that $P \neq \emptyset$ and terminate.

  (b) Otherwise, let $a_i \in A$, $b_i \in b$ be a violated constraint, i.e., $\langle a_i\, ,\, c_t\rangle > b_i$; let $E_{t+1} = E(c_t, Q_t)$ be the smallest volume ellipsoid containing the half-ellipsoid $E_t \cap \{x \mid \langle a_i\, ,\, x\rangle \leqslant \langle a_i\, ,\, c_t\rangle\}$.[a]

[a]This step is still unspecified as we did not mention *how* to compute this ellipsoid E_{t+1} exactly; we present a formula for this step in Lemma 12 which concludes the description of this algorithm.

The analysis of the Ellipsoid algorithm relies primarily on the following result.

Proposition 9. *Let $E(c, Q)$ by any ellipsoid in $\mathbb{R}^n$ and E' be the smallest volume ellipsoid that contains the half-ellipsoid $E \cap \{x \mid \langle a, x\rangle \leqslant \langle a, c\rangle\}$ for any arbitrary $a \in \mathbb{R}^n$. Then,*

$$\text{Volume}(E') \leqslant \left(1 - \frac{1}{5n}\right) \cdot \text{Volume}(E). \tag{2}$$

We postpone more details on this proposition and in particular how to compute E' to the next section. Here, we show that how this result can be used to conclude the proof of correctness of the Ellipsoid algorithm.

Theorem 10. *The Ellipsoid algorithm, given any polytope (P) defined as $\{x \in \mathbb{R}^n \mid Ax \leqslant b\}$ under Assumption 8, correctly decides whether (P) is empty and if not outputs a point $x \in P$ in $O(n^2 \cdot \ln(\frac{R}{r}))$ iterations.*

Proof. By our assumption, $P \subseteq E_1$ and by the definition of algorithm, we have $P \subseteq E_t$ for every iteration t as well. Moreover, the algorithm only returns P is non-empty if it ever finds a point inside P, and only ever returns P is empty, if its while-loop terminates. Moreover, since $P \subseteq E_t$, if the while-loop terminates, we have that $\text{Volume}(P) < \text{Volume}(B(0, r))$ which by Assumption 8 means P is empty. Thus, we only need to show that when P is empty, the algorithm indeed breaks from its while-loop.

To do so, we just have to calculate what happens to the volume of the ellipsoid across iterations and then check when this becomes so small that it violates the while-loop condition.

Let t be the number of steps taken so far. We have,

$$\text{Volume}(E_{t+1}) \leqslant \left(1 - \frac{1}{5n}\right) \cdot \text{Volume}(E_t) \leqslant \left(1 - \frac{1}{5n}\right)^t \cdot \text{Volume}(E_1) = \left(1 - \frac{1}{5n}\right)^t \cdot \text{Volume}(B(0, R)),$$

where the first inequality is by Proposition 9, the second is by applying the first inductively on each prior iteration, and the final equality is by the choice of E_1.

[3]In the next lecture, we specify how to satisfy the conditions of this assumption when solving any LP feasibility problem.

For $t > 5n^2 \ln(\frac{R}{r})$, by the above inequality, we have,

$$
\begin{aligned}
\text{Volume}(E_{t+1}) &\leqslant \left(1 - \frac{1}{5n}\right)^t \cdot R^n \cdot \text{Volume}(B(0,1)) && \text{(by Fact 6 as } B(0,R) = \{(R \cdot I) \cdot x \mid x \in B(0,1)\}) \\
&\leqslant \exp\left(-\frac{t}{5n}\right) \cdot R^n \cdot \text{Volume}(B(0,1)) && \text{(as } 1 - z \leqslant \exp(-z) \text{ for all } z \in (0,1)) \\
&< \exp\left(-n \ln(\frac{R}{r})\right) R^n \cdot \text{Volume}(B(0,1)) && \text{(by the choice of } t > 5n^2 \ln(\tfrac{R}{r})) \\
&= R^{-n} \cdot r^n \cdot R^n \cdot \text{Volume}(B(0,1)) \\
&= \text{Volume}(B(0,r)). && \text{(by Fact 6 exactly as in the first equation)}
\end{aligned}
$$

This implies that after these many iterations, $\text{Volume}(E_{t+1}) < \text{Volume}(B(0,r))$, thus the while-loop terminates. This concludes the proof. □

3 Computing the Smallest Volume Ellipsoid

Recall that in Line Item *(ii)*b of the ellipsoid algorithm, we need to compute the smallest ellipsoid containing the half-ellipsoid to one side of a separating hyperplane. This section describes how to implement this step. We begin by considering a special, easier case first, and then sketch the idea behind solving the general case by reducing it to the easy case.

3.1 Easy Case

Suppose we have $E_0 = B(0,1)$ and $a_0 = (-1, 0, ..., 0)$ is our separating hyperplane among constraints $Ax \leqslant b$. Our goal is to find an ellipsoid E_1 that contains the region

$$T = B(0,1) \cap \{x \mid \langle a_0 \,,\, x\rangle \leqslant \langle a_0 \,,\, 0\rangle\} = B(0,1) \cap \{x \mid x_1 \geqslant 0\}.$$

Intuitively, the new ellipsoid should have a center slightly above the previous one and should have some vertical contraction and horizontal expansion as seen in Figure 2 below.

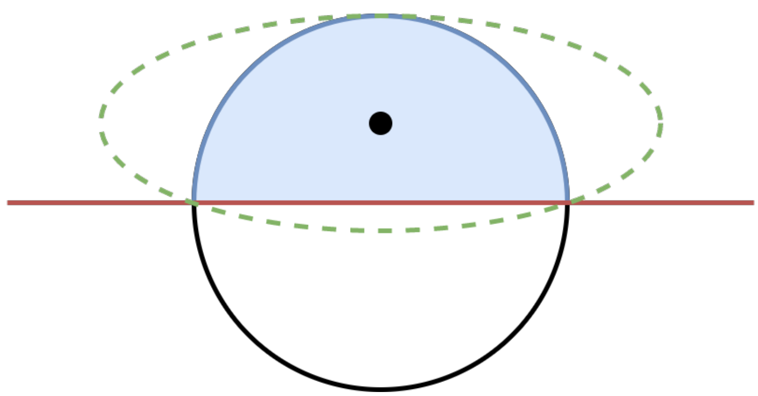

Figure 2: The smallest ellipsoid containing the upper half of $B(0,1)$.

The following lemma specifies how to pick E_1 and what is its resulting volume.

Lemma 11. *Suppose $n \geqslant 10$ and define $c_1 = \left(\frac{1}{n+1}, 0, ..., 0\right)$ and*

$$Q_1 = \frac{n^2}{n^2-1} \begin{bmatrix} \frac{n-1}{n+1} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 1 \end{bmatrix}.$$

Then, $E_1 = E(c_1, Q_1)$ contains the half-ball $T := B(0,1) \cap \{x \mid x_1 \geqslant 0\}$ and

$$\text{Volume}(E_1) \leqslant \left(1 - \frac{1}{5n}\right) \cdot \text{Volume}(E_0).$$

Before getting to the proof, an important remark is in order.

Remark. In Lemma 11, we are actually not going to prove E_1 is the minimum volume ellipsoid (even though that statement is also true). But notice that the conclusion of the lemma on the volume and that it contains the feasible region is sufficient for using this result (and its generalization) directly in the proof of Theorem 10.

It is however worth mentioning that the choices of c_1 and Q_1 do not come out of nowhere (although they may look like it given just the statement of the lemma) but in fact are the results of a natural optimization problem for minimizing the volume subject to the constraint of containing the half-ball T. We will not explore this step in this lecture as it is beyond the scope of this course.

Finally, note that for simplicity of exposition, we assumed n is sufficiently large (i.e., $n \geqslant 10$) in this lemma. This is only for the simplicity of our calculations as it does not affect the asymptotic bounds we get on the performance of the Ellipsoid algorithm (say, by artificially increasing the dimension from $n < 10$ to 10 by introducing free variables). Alternatively, one can also simply work with slightly weaker bounds on the volume of the ellipsoid that can be proven without this assumption and everything works out as before in the proof of Theorem 10.

Proof of Lemma 11. The first step is to show that any point $x \in T$ also belongs to E_1. I.e., show that for any $x \in T$,

$$(x - c_1)^\top Q_1^{-1} \cdot (x - c_1) \leqslant 1,$$

and thus $x \in E_1$.

Let $x = (x_1, x_2, ..., x_n) = (x_1, x_{-1})$, where $x_{-1} = (x_2, x_3, ..., x_n)$. Given that $x \in T$ we have the following equations:

$$\|x\| \leqslant 1, \qquad 0 \leqslant x_1 \leqslant 1, \qquad \text{and} \qquad x_1^2 + \|x_{-1}\|^2 \leqslant 1. \tag{3}$$

We use this in the following as follows:

$$\begin{aligned}
(x - c_1)^\top Q_1^{-1}(x - c_1) &= \frac{n^2 - 1}{n^2} \cdot [(x_1 - \frac{1}{1+n}), x_{-1}] \cdot \begin{bmatrix} \frac{n-1}{n+1} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 1 \end{bmatrix} \cdot [(x_1 - \frac{1}{1+n}), x_{-1}]^\top \\
&\qquad\qquad \text{(by the definition of } c_1 \text{ and } Q_1\text{)} \\
&= \frac{n^2 - 1}{n^2} \cdot \frac{n+1}{n-1} \cdot \left(x_1 - \frac{1}{1+n}\right)^2 + \frac{n^2 - 1}{n^2} \cdot \|x_{-1}\|^2 \\
&\qquad\qquad \text{(by expanding the terms and using } x_{-1}^\top \cdot x_{-1} = \|x_{-1}\|^2\text{)} \\
&= \frac{1}{n^2}\left((n+1) \cdot x_1 - 1\right)^2 + \frac{n^2 - 1}{n^2} \cdot \|x_{-1}\|^2 \\
&\leqslant \frac{1}{n^2}\left((n+1) \cdot x_1 - 1\right)^2 + \frac{n^2 - 1}{n^2} \cdot (1 - x_1^2) \qquad \text{(by Eq (3), } \|x_{-1}\|^2 \leqslant (1 - x_1^2)\text{)} \\
&= \frac{(n+1)^2}{n^2} \cdot x_1^2 - 2 \cdot \frac{(n+1)}{n^2} \cdot x_1 + \frac{1}{n^2} + \frac{n^2 - 1}{n^2} - \frac{n^2 - 1}{n^2} \cdot x_1^2 \\
&\qquad\qquad \text{(by expanding the terms)} \\
&= \frac{2 \cdot (n+1)}{n^2} \cdot (x_1^2 - x_1) + 1 \\
&\leqslant 1. \qquad\qquad \text{(by Eq (3), } x_1^2 - x_1 \leqslant 0\text{)}
\end{aligned}$$

Hence, x is in the ellipsoid E_1, proving that $T \subseteq E_1$.

We now bound the volume of E_1. Since $E_0 = B(0,1)$ and by Proposition 9,

$$\begin{aligned}
\frac{\mathrm{Vol}(E_1)}{\mathrm{Vol}(E_0)} &= \sqrt{\det(Q)} \\
&= \sqrt{\left(1 - \frac{2}{n+1}\right)\left(1 + \frac{1}{n^2-1}\right)^n} \\
&\qquad\qquad \text{(as } Q \text{ is a diagonal matrix and the numbers on its main diagonal)} \\
&\leqslant \sqrt{\exp\left(-\frac{2}{n+1} + \frac{n}{n^2-1}\right)} \qquad \text{(as } 1 - z \leqslant \exp(-z) \text{ and } (1+z) \leqslant \exp(z) \text{ for } z \in (0,1)) \\
&= \exp\left(\frac{1}{2} \cdot \left(\frac{-2n+2+n}{n^2-1}\right)\right) \qquad \text{(as } n^2 - 1 = (n-1)\cdot(n+1)) \\
&= \exp\left(-\left(\frac{n-2}{2n^2-2}\right)\right) \\
&\leqslant 1 - \frac{n-2}{4n^2-4} \qquad \text{(as } \exp(-z) \leqslant 1 - z/2 \text{ for } z \in (0,1/2)) \\
&\leqslant 1 - \frac{1}{5n}, \qquad \text{(for } n \geqslant 10)
\end{aligned}$$

concluding the proof. □

3.2 General Case

We now consider the general case of Line (*ii*)b. Suppose we have an ellipsoid $E_t = E(c_t, Q_t)$ and a separating hyperplane $\langle a_t\,,\, x - c_t\rangle = 0$. Our goal is to compute the minimum volume ellipsoid E_{t+1} that contains the half-ellipsoid bounded by $E_t \cap \{x \mid \langle a_t\,,\, x\rangle \leqslant \langle a_t\,,\, c_t\rangle\}$.

Lemma 12. *Given an ellipsoid $E_t = E(c_t, Q_t)$ and a separating hyperplane $\langle a_t\,,\, x - c_t\rangle = 0$, define the ellipsoid $E_{t+1} = E(c_{t+1}, Q_{t+1})$ as:*

$$\begin{aligned}
c_{t+1} &= c_t - \frac{h_t}{n+1}, \\
Q_{t+1} &= \frac{n^2}{n^2-1} \cdot (Q_t - \frac{2}{n+1} \cdot h_t \cdot h_t^\top),
\end{aligned}$$

where h_t is the unit vector

$$h_t := \frac{Q_t \cdot a_t}{\sqrt{a_t^\top \cdot Q_t \cdot a_t}}.$$

Then, if $n \geqslant 10$, E_{t+1} contains the half-ellipsoid $T := E_t \cap \{x \mid \langle a_t\,,\, x\rangle \leqslant \langle a_t\,,\, c_t\rangle\}$ and

$$\mathrm{Volume}(E_{t+1}) \leqslant \left(1 - \frac{1}{5n}\right) \cdot \mathrm{Volume}(E_t).$$

We will not prove this lemma in this course and only take it for granted. We only note that this proof is by showing that there is an invertible affine map $L : \mathbb{R}^n \to \mathbb{R}^n$ that when applied to E_t and T, reduces this case to the easy case we already examined in Lemma 11. I.e.,

$$L(E_t) = B(0,1), \qquad \text{and} \qquad L(\{x \mid \langle a_t\,,\, x\rangle \leqslant \langle a_t\,,\, c_t\rangle\}) = \{x \mid x_1 \geqslant 0\}\,.$$

Thus, we can take E' be the ellipsoid of Lemma 11 and we get $E_{t+1} = L^{-1}(E')$. The bound on the volume then is as follows:

$$\begin{aligned}
\frac{\mathrm{Volume}(E_{t+1})}{\mathrm{Volume}(E_t)} &= \frac{\mathrm{Volume}(L^{-1}(E'))}{\mathrm{Volume}(L^{-1}(B(0,1)))} \\
&= \frac{\det(L^{-1}) \cdot \mathrm{Volume}(E')}{\det(L^{-1}) \cdot \mathrm{Volume}(B(0,1))} && \text{(by Fact 6)} \\
&\leqslant \left(1 - \frac{1}{5n}\right). && \text{(by Lemma 11)}
\end{aligned}$$

This concludes our description of the Ellipsoid algorithm. In particular, we implement Line *(ii)*b of the algorithm using Lemma 12 in polynomial time by computing c_{t+1} and Q_{t+1} using matrix multiplication formulas of the lemma. However, we need to mention the following remark in this context.

Remark. As can be seen from Lemma 12, running Line *(ii)*b of the Ellipsoid algorithm requires taking *square roots*. This is quite problematic in general because one can in general cannot take square root of numbers in polynomial time (simply because their description may even be unbounded, say taking $\sqrt{2}$ which is an irrational number). Thus, when implementing this algorithm, we need to compute square roots approximately to within a bounded precision. One can prove that taking the precision to be quite small, although still polynomial in the input size, is sufficient for making all the steps of the algorithm to go through, and obtain that every iteration of the Ellipsoid algorithm can be implemented in polynomial time. This part requires a quite tedious proof and we do not cover it in this course.

In the next lecture, we will show how to lift Assumption 8, i.e., show that there is always a choice of R and r which are not too large (the latter requires "relaxing" the polytope a bit) and obtain a polynomial time algorithm for solving LPs.